On Hilbert's tenth problem over subrings of $\mathbb{Q}$

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Abstract

For a subring R of the rationals $\mathbb{Q}$, Hilbert's tenth problem over R, denoted by $\mathrm{HTP}(R)$, is the set of multivariate polynomial equations with integer coefficients which has a solution in R. The undecidability status of $\mathrm{HTP}(\mathbb{Q})$ is a big open problem in both computability theory and number theory. Although it is possible that $\mathrm{HTP}(\mathbb{Q})$ has incomplete Turing degree, no attempt has been made so far to characterize the undecidability of $\mathrm{HTP}(\mathbb{Q})$. In this thesis, we introduce a new notion called HTP-nontriviality of a subring of $\mathbb{Q}$, and show that the following conditions are equivalent: (1) $\mathrm{HTP}(\mathbb{Q})$ is undecidable, (2) there are nonmeager many HTP-nontrivial subrings of $\mathbb{Q}$ and (3) Player I has a winning strategy for the Banach-Mazur game for the set of HTP-nontrivial subrings. We also show a measure-theoretic analogue of this result.

1 Introduction

Hilbert's tenth problem over a ring R, denoted by $\mathrm{HTP}(R)$, is a decision problem as to whether a given diophantine equation has a solution in R. Originally, Hilbert asked the case where $R = \mathbb{Z}$, the ring of rational integers. In 1970, Matiyasevich [Mat70] proved that $\mathrm{HTP}(\mathbb{Z})$ is undecidable, based on the works by Robison, Davis and Putnam [DPR61]. More precisely, they showed that the Halting Problem $\emptyset'$ is 1-reducible to $\mathrm{HTP}(\mathbb{Z})$.

In contrast to the case $R = \mathbb{Z}$, the undecidability of $\mathrm{HTP}(\mathbb{Q})$, where $\mathbb{Q}$ denotes the field of rational numbers, still remains a big open problem. For a typical ring R, a standard way to prove the undecidability of $\mathrm{HTP}(R)$ is to show that $\mathbb{Z}$ admit a diophantine model in R. Here a *diophantine model* of $\mathbb{Z}$ over R is a subset $A \subseteq R^n$ for some n which is diophantine over R, equipped with a bijection $\mathbb{Z} \to A$ under which the image of the graphs of addition and multiplication on $\mathbb{Z}$ are also diophantine as subsets of R^{3n}. A subset $D \subseteq R^n$ is called *diophantine* if D is the projection of the zero locus of some polynomial, i.e., there exists a polynomial $f(\vec{a}, \vec{x}) \in R[\vec{a}, \vec{x}]$ in $n+m$ variables for some m such that $D = \{\, \vec{a} \in R^n \mid \exists \vec{x} \in R^m [f(\vec{a}, \vec{x}) = 0] \,\}$. If $\mathbb{Z}$ admit a diophantine model in R, then $\mathrm{HTP}(\mathbb{Z})$ is 1-reducible to $\mathrm{HTP}(R)$ and $\emptyset'$ is thus 1-reducible to $\mathrm{HTP}(R)$. However, Cornelissen and Zahidi [CZ00] showed that $\mathbb{Z}$ does not admit a diophantine model in $\mathbb{Q}$ under Mazur's conjecture [Maz92, Conjecture 3]. Therefore it is necessary to develop another technique to prove the undecidability (or possibly decidability) of $\mathrm{HTP}(\mathbb{Q})$ unless Mazur's conjecture is false.

On the other hand, there is another important open problem related to the undecidable problems: is there a "natural" undecidable problem whose Turing degree is incomplete? A noncomputable set A of natural numbers has incomplete Turing degree if $A <_{\mathrm{T}} \emptyset'$. Friedberg [Fri57] and Mučnik [Muc56] independently showed that there indeed exists a computably enumerable set A of natural numbers such that $\emptyset <_{\mathrm{T}} A <_{\mathrm{T}} \emptyset'$. However, the set A they constructed is highly artificial, and no "natural" example of undecidable problem with incomplete Turing degree is known so far. Moreover, for every "natural" decision problem A known to be undecidable, its undecidability is proved by constructing a Turing reduction from $\emptyset'$ to A.

It is possible that the undecidability of $\mathrm{HTP}(\mathbb{Q})$ might be proved without constructing a Turing reduction from $\emptyset'$. Moreover, it is also possible that $\mathrm{HTP}(\mathbb{Q})$ might have incomplete Turing degree. The first result in this direction is Theorem 1.1 below, which is given by Miller. From now on, we freely use terminologies and notations defined in Section 2.

Theorem 1.1 (Miller, Corollary 1 in [Mil16])**.** *Let* $\mathbb{P} := \{2, 3, 5, 7, \dots\}$ *be the set of prime numbers. For* *every set $C \in 2^\omega$, the following conditions are equivalent.*

1. $C \leq_{\mathrm{T}} \mathrm{HTP}(\mathbb{Q})$.
2. $\{ W \in 2^{\mathbb{P}} \mid C \leq_{\mathrm{T}} \mathrm{HTP}(R_W) \} = 2^{\mathbb{P}}$.
3. $\{ W \in 2^{\mathbb{P}} \mid C \leq_{\mathrm{T}} \mathrm{HTP}(R_W) \}$ *is not meager in* $2^{\mathbb{P}}$.

However, Theorem 1.1 has a crucial problem: if $\mathrm{HTP}(\mathbb{Q})$ had incomplete Turing degree, how could one define an undecidable set $C \leq_{\mathrm{T}} \mathrm{HTP}(\mathbb{Q})$? One option is setting $C = \emptyset'$, whereas there are only meager many set W known to satisfy $\emptyset' \leq_{\mathrm{T}} \mathrm{HTP}(R_W)$. We discuss this point again in Section 3.1.

Although the possibility that $\mathrm{HTP}(\mathbb{Q})$ has incomplete Turing degree has been recognized, no attempt has been made so far to characterize the undecidability of $\mathrm{HTP}(\mathbb{Q})$. In this thesis, we introduce a new notion, HTP-nontriviality of subrings of $\mathbb{Q}$ (Definition 3.1). Then we show the following theorem, which states that the undecidability of $\mathrm{HTP}(\mathbb{Q})$ can be characterized as the abundance of the HTP-nontrivial subrings and the existence of a winning strategy for certain Banach-Mazur game.

Theorem (Theorem 3.2)**.** *The following conditions are equivalent.*

1. $\mathrm{HTP}(\mathbb{Q}) >_{\mathrm{T}} \emptyset$, *i.e.,* $\mathrm{HTP}(\mathbb{Q})$ *is undecidable.*
2. *There are comeager many HTP-nontrivial subrings of* $\mathbb{Q}$.
3. *There are nonmeager many HTP-nontrivial subrings of* $\mathbb{Q}$.
4. *There is a subring of* $\mathbb{Q}$ *which is both HTP-nontrivial and HTP-generic.*
5. *Player* I *has a winning strategy for Banach-Mazur game for the HTP-nontrivial subrings of* $\mathbb{Q}$.
6. *Player* II *does not have a winning strategy for Banach-Mazur game for the HTP-nontrivial subrings of* $\mathbb{Q}$.

Theorem 3.2 opens a new avenue for undecidability proofs for $\mathrm{HTP}(\mathbb{Q})$. That is, the undecidability of $\mathrm{HTP}(\mathbb{Q})$ reduces to the problem to find a winning strategy for certain game. The most important point is that undecidability proofs based on Theorem 3.2 also work even if $\mathrm{HTP}(\mathbb{Q})$ has incomplete Turing degree and no fixed noncomputable set C is needed in contrast to Miller's Theorem 1.1. Moreover, if one find such proof, then $\mathrm{HTP}(\mathbb{Q})$ become the first undecidable problem whose undecidability is established without constructing a Turing reduction from the Halting Problem $\emptyset'$.

We also show a measure-theoretic analogue of this theorem (Theorem 3.4). In Section 3.2, we reinterpret a known construction in terms of Banach-Mazur game. In Section 3.3, we show some partial results about abundance of HTP-nontrivial subsets.

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2 Preliminaries

In this thesis, we consider $\mathrm{HTP}(R)$ only for subrings R of $\mathbb{Q}$ to approach $\mathrm{HTP}(\mathbb{Q})$. Let $\mathbb{P} := \{2, 3, 5, 7, \dots\}$ be the set of prime numbers. Every subring of $\mathbb{Q}$ is of the form $R_W := \mathbb{Z}[W^{-1}]$ for some subset W of $\mathbb{P}$. The function $W \mapsto R_W$ is a bijection from the power set $2^{\mathbb{P}}$ of $\mathbb{P}$ to the set of subrings of $\mathbb{Q}$. Define p_n as the n-th prime number, starting with $p_0 = 2$. The function $n \mapsto p_n$ is a bijection from the set $\omega = \{0, 1, 2, \dots\}$ of natural numbers to $\mathbb{P}$. So the Turing degree of a subset W of $\mathbb{P}$ is that of W as a subset of ω.

Let $W \in 2^{\mathbb{P}}$. For simplicity, we regard $\mathrm{HTP}(R_W)$ as the set of polynomials with integer coefficients which have a solution in R_W, that is,

$$\mathrm{HTP}(R_W) = \{ f(x_1, \dots, x_n) \in \mathbb{Z}[x_1, x_2, \dots] \mid \exists \vec{x} \in (R_W)^n [f(\vec{x}) = 0] \}.$$

Here, considering the polynomials in $\mathbb{Z}[x_1, x_2, \dots]$ instead of $R_W[x_1, x_2, \dots]$ does not make any difference since one can eliminate denominators which appears in polynomials in $R_W[x_1, x_2, \dots]$. By fixing a computable bijection $\omega \to \mathbb{Z}[x_1, x_2, \dots]$, we regard $\mathrm{HTP}(R_W)$ as a subset of ω. So the Turing degree of $\mathrm{HTP}(R_W)$ also make sense.

2.1 Topology and measure on the Cantor space $2^{\mathbb{P}}$

The Cantor space 2^ω is the power set of ω. Because of the standard bijection $\omega \ni n \mapsto p_n \in \mathbb{P}$, we also call $2^{\mathbb{P}}$ Cantor space. We often view $2^{\mathbb{P}}$ as the set of infinite paths through the complete binary tree $2^{<\mathbb{P}}$. An element of $2^{<\mathbb{P}}$ can be regarded as a binary string of finite length, i.e., a function from a downward closed finite subset of $\mathbb{P}$ to $\{0,1\}$. An element of $2^{\mathbb{P}}$ also can be regarded as a binary string of infinite length. For two binary strings σ and τ in $2^{<\mathbb{P}}$, we write $\sigma \preccurlyeq \tau$ if σ is an initial segment of τ; $\sigma \prec \tau$ means that $\sigma \preccurlyeq \tau$ and $\sigma \neq \tau$. Since a set $W \in 2^{\mathbb{P}}$ can be viewed as an infinite sequence, the notation $\sigma \prec W$ also makes sense. The concatenation of two strings σ and τ in $2^{<\mathbb{P}}$ is denoted by $\sigma^\frown\tau$. The length of a string $\sigma \in 2^{<\mathbb{P}}$ is denoted by $|\sigma|$. For a set $W \in 2^{\mathbb{P}}$ and $n \in \omega$, $W{\upharpoonright}n$ denotes the initial segment of W which is of length n. For $i \in \{0,1\}$ and $n \in \omega \cup \{\infty\}$, i^n denotes the string of length n which only consists of i.

A basic open set of $2^{\mathbb{P}}$ is given in the form

$$\mathcal{U}_\sigma := \{\, W \in 2^{\mathbb{P}} \mid \sigma \prec W \,\}$$

for some binary string $\sigma \in 2^{<\mathbb{P}}$. The topology on $2^{\mathbb{P}}$ generated by $\{\mathcal{U}_\sigma \mid \sigma \in 2^{<\mathbb{P}}\}$ coincide with the topology on $2^{\mathbb{P}}$ given by the countable product topology of the discrete space $\{0,1\}$. A subset $\mathcal{X} \subseteq 2^{\mathbb{P}}$ is called *nowhere dense* if interior of its closure $\mathrm{Int}(\mathrm{Cl}(\mathcal{X}))$ is the empty set. The union of a countable family of nowhere dense sets is called *meager*. The complement of meager set in $2^{\mathbb{P}}$ is called *comeager*. From the fact that the Cantor space $2^{\mathbb{P}}$ is completely metrizable, Baire category theorem ensures that every comeager set in $2^{\mathbb{P}}$ is not meager.

The Cantor space $2^{\mathbb{P}}$ can be equipped with the standard probability measure μ, which satisfies $\mu(\mathcal{U}_\sigma) = 2^{-|\sigma|}$ for any binary string $\sigma \in 2^{<\mathbb{P}}$. In particular, $\mu(2^{\mathbb{P}}) = \mu(\mathcal{U}_\emptyset) = 1$. A subset $\mathcal{X} \subseteq 2^{\mathbb{P}}$ is called *null* if $\mu(\mathcal{X}) = 0$.

2.2 Some properties of the HTP-operator

Proposition 2.1. *For any set $W \in 2^{\mathbb{P}}$, $\mathrm{HTP}(R_W)$ is c.e. in W and*

$$W \oplus \mathrm{HTP}(\mathbb{Q}) \leq_1 \mathrm{HTP}(R_W) \leq_1 W'$$

where $\oplus$ denotes the Turing join and W' denotes the Turing jump of W.

Proof. The reduction $W \leq_1 \mathrm{HTP}(R_W)$ is established by the injection $p \mapsto px - 1$. For $\mathrm{HTP}(\mathbb{Q}) \leq_1 \mathrm{HTP}(R_W)$, see Corollary 5.2 in [EMPS17]. The rest part is easy. □

Definition 2.2 (cf. Section 3 in [Mil17])**.** For every polynomial $f \in \mathbb{Z}[x_1, x_2, \dots]$, define

$$\begin{aligned}
\mathcal{A}(f) &:= \{\, W \in 2^{\mathbb{P}} \mid f \in \mathrm{HTP}(R_W) \,\},\\
\mathcal{B}(f) &:= \partial\mathcal{A}(f), \text{ the boundary of } \mathcal{A}(f),\\
\mathcal{C}(f) &:= \mathrm{Int}(\overline{\mathcal{A}(f)}), \text{ the exterior of } \mathcal{A}(f).
\end{aligned}$$

Lemma 2.3. *For every polynomial $f \in \mathbb{Z}[x_1, x_2, \dots]$, $\mathcal{A}(f)$ is an open set in $2^{\mathbb{P}}$. In particular, $\mathcal{B}(f)$ is a nowhere dense set in $2^{\mathbb{P}}$.*

Proof. Let $f \in \mathbb{Z}[x_1, x_2, \dots]$ be a polynomial in n variable and $W \in \mathcal{A}(f)$. Then $f \in \mathrm{HTP}(R_W)$ and there is a solution $\vec{x} \in (R_W)^n$ such that $f(\vec{x}) = 0$. Since $R_W = \bigcup_{m\in\omega} R_{(W\upharpoonright m)^\frown 0^\infty}$, there exists some $m \in \omega$ such that $\vec{x} \in (R_{(W\upharpoonright m)^\frown 0^\infty})^n$. Then for any set $V \succ W{\upharpoonright}m$ satisfies $\vec{x} \in (R_V)^n$ and we obtain $\mathcal{U}_{W\upharpoonright m} \subseteq \mathcal{A}(f)$.

In general, the boundary set of an open set is closed and has no interior point. Therefore $\mathcal{B}(f)$ is nowhere dense in $2^{\mathbb{P}}$. □

Definition 2.4 (Definition 2 in [Mil16])**.** The *entire boundary set* is defined as $\mathcal{B} := \bigcup_{f\in\mathbb{Z}[x_1,x_2,\dots]} \mathcal{B}(f)$, which is a meager set by Lemma 2.3. A set $W \in 2^{\mathbb{P}}$ is called *HTP-generic* if $W \notin \mathcal{B}$. HTP-genericity is comeager.

However, the measure of the entire boundary set $\mathcal{B}$ is unknown.

Proposition 2.5 (Miller, Proposition 2 in [Mil16])**.** *For every HTP-generic set $W \in 2^{\mathbb{P}}$,*

$$\mathrm{HTP}(R_W) \leq_{\mathrm{T}} W \oplus \mathrm{HTP}(\mathbb{Q}).$$

Note that the opposite reduction $\geq_{\mathrm{T}}$ holds for any $W \in 2^{\mathbb{P}}$.

2.3 Banach-Mazur game

Banach-Mazur game is a two-player infinite game introduced by S. Mazur. We only use a special version of Banach-Mazur game, which is for the Cantor space 2^ω.

For a subset $\mathcal{A}$ of 2^ω, the *Banach-Mazur game* for $\mathcal{A}$, denoted by $\mathrm{BM}(\mathcal{A})$, is defined as follows. First, Player I chooses a binary string $\sigma_0 \in 2^{<\omega}$. Then Player II chooses $\sigma_1 \in 2^{<\omega}$ such that $\sigma_0 \prec \sigma_1$. For given $\sigma_{2i-1} \in 2^{<\omega}$ ($i \geq 1$), Player I chooses $\sigma_{2i} \in 2^{<\omega}$ such that $\sigma_{2i-1} \prec \sigma_{2i}$ and then Player II chooses $\sigma_{2i+1} \in 2^{<\omega}$ such that $\sigma_{2i} \prec \sigma_{2i+1}$. Player I wins $\mathrm{BM}(\mathcal{A})$ if the resulting point $A = \bigcup_{n\in\omega} \sigma_n$ of 2^ω belongs to $\mathcal{A}$. Player II wins otherwise.

Definition 2.6. A *strategy* for Player I is a map $\alpha\colon \bigcup_{i\in\omega}(2^{<\omega})^{2i} \to 2^{<\omega}$ such that $\alpha(\sigma_0, \dots, \sigma_{2i-1}) \succ \sigma_{2i-1}$ for any finite sequence $(\sigma_0, \dots, \sigma_{2i-1})$ of even length. Similarly, a strategy for Player II is also defined as a map $\beta\colon \bigcup_{i\in\omega}(2^{<\omega})^{2i+1} \to 2^{<\omega}$. The *play* $\alpha * \beta$ for strategies α and β is the resulting point $A = \bigcup_{n\in\omega} \sigma_n$ when Player I follows α and Player II follows β. A strategy α for Player I is called a *winning strategy* for $\mathrm{BM}(\mathcal{A})$ if $\alpha * \beta \in \mathcal{A}$ for any strategy β for Player II. Similarly, a strategy β for Player II is a winning strategy for $\mathrm{BM}(\mathcal{A})$ if $\alpha * \beta \notin \mathcal{A}$ for any strategy α for Player I.

Lemma 2.7. *For any subset $\mathcal{A} \subseteq 2^\omega$, the following properties hold.*

1. *If $\mathcal{A}$ is comeager, then Player I has a winning strategy for $\mathrm{BM}(\mathcal{A})$.*
2. *If $\mathcal{A}$ is meager, then Player II has a winning strategy for $\mathrm{BM}(\mathcal{A})$.*

Proof. Suppose $\mathcal{A}$ is comeager. From the definition of meagerness, there exists a family $(\mathcal{X}_s)_{s\in\omega}$ of nowhere dense sets such that $\overline{\mathcal{A}} = \bigcup_{s\in\omega} \mathcal{X}_s$. Let σ_{2s-1} be the last move of Player II where $\sigma_{-1} = \emptyset$. Since the closure $\mathrm{Cl}(\mathcal{X}_s)$ has no interior point, $\mathcal{U}_{\sigma_{2s-1}} - \mathrm{Cl}(\mathcal{X}_s)$ is a nonempty open set of 2^ω. Hence Player I can choose $\sigma_{2s} \succ \sigma_{2s-1}$ so that $\mathcal{U}_{\sigma_{2s}} \cap \mathrm{Cl}(\mathcal{X}_s) = \emptyset$. Then the resulting set $A = \bigcup_{n\in\omega} \sigma_n$ satisfies $A \in \bigcap_{s\in\omega} \overline{\mathcal{X}_s} = \mathcal{A}$. The rest half of proof is similar. $\square$

In fact, Player II has a winning strategy for $\mathrm{BM}(\mathcal{A})$ if $\mathcal{A}$ is meager. For detail, see e.g. 6A.14 in [Mos09].

3 Results

3.1 Equivalent conditions for the undecidability of $\mathrm{HTP}(\mathbb{Q})$

Definition 3.1. We say that a set $W \in 2^{\mathbb{P}}$ is *HTP-nontrivial* if $W <_{\mathrm{T}} \mathrm{HTP}(R_W)$. We write $\mathcal{N}$ for the set of the HTP-nontrivial sets. Similarly, the *1-nontrivial*, *m-nontrivial*, *tt-nontrivial* and *bT-nontrivial* sets are defined by replacing Turing reduction $<_{\mathrm{T}}$ in the above definition by 1-reduction $<_1$, many-one reduction $<_{\mathrm{m}}$, truth-table reduction $<_{\mathrm{tt}}$ and bounded Turing reduction (also known as weak truth-table reduction) $<_{\mathrm{bT}}$, respectively. For $r \in \{1, \mathrm{m}, \mathrm{tt}, \mathrm{bT}\}$, We write $\mathcal{N}_r$ for the set of the r-nontrivial sets.

Theorem 3.2. *The following conditions are equivalent.*

1. $\mathrm{HTP}(\mathbb{Q}) >_{\mathrm{T}} \emptyset$, *i.e.,* $\mathrm{HTP}(\mathbb{Q})$ *is undecidable.*
2. *$\mathcal{N}$ is comeager in $2^{\mathbb{P}}$.*
3. *$\mathcal{N}$ is not meager in $2^{\mathbb{P}}$.*

4. $\mathcal{N} \cap \overline{\mathcal{B}} \neq \emptyset$.
5. Player I *has a winning strategy for* BM($\mathcal{N}$).
6. Player II *does not have a winning strategy for* BM($\mathcal{N}$).

In particular, BM($\mathcal{N}$) *is determined, i.e., either Player* I *or* II *has a winning strategy for* BM($\mathcal{N}$).

Proof of Theorem. (1) $\Longrightarrow$ (2). It is well-known that the Turing upper cone $\{ B \in 2^\omega \mid A \leq_{\mathrm{T}} B \}$ over a set $A \in 2^\omega$ is meager in 2^ω unless A is computable (for example, see EXAMPLE 13.2.21 in [Coo04]). Therefore there are comeager many sets in which HTP($\mathbb{Q}$) is not computable. By Proposition 2.1, every such set W satisfies $W \not\geq_{\mathrm{T}} \mathrm{HTP}(\mathbb{Q}) \leq_{\mathrm{T}} \mathrm{HTP}(R_W)$ and thus $W <_{\mathrm{T}} \mathrm{HTP}(R_W)$.

(2) $\Longrightarrow$ (3) $\Longrightarrow$ (4). Trivial.

(4) $\Longrightarrow$ (1). Let $W \in \mathcal{N} \cap \overline{\mathcal{B}}$. Since W is both HTP-nontrivial and HTP-generic, it follows from Proposition 2.5 that

$$W <_{\mathrm{T}} \mathrm{HTP}(R_W) \leq_{\mathrm{T}} W \oplus \mathrm{HTP}(\mathbb{Q}). \qquad (*)$$

Hence HTP($\mathbb{Q}$) must be undecidable.

(2) $\Longrightarrow$ (5) $\Longrightarrow$ (6) $\Longrightarrow$ (3). By Lemma 2.7. □

Since $\overline{\mathcal{B}}$ is comeager, we may use BM($\mathcal{N} \cap \overline{\mathcal{B}}$) instead of BM($\mathcal{N}$) in Theorem 3.2. Theorem 3.2 implies that $\mathcal{N}$ (resp. $\mathcal{N} \cap \overline{\mathcal{B}}$) is either meager (resp. empty) or comeager, therefore it also can be viewed as a topological zero-one law for $\mathcal{N}$ (resp. $\mathcal{N} \cap \overline{\mathcal{B}}$).

Corollary 3.3. *The following conditions are equivalent.*

1. HTP($\mathbb{Q}$) *is decidable.*
2. $\mathcal{N}$ *is not comeager in* $2^{\mathbb{P}}$.
3. $\mathcal{N}$ *is meager in* $2^{\mathbb{P}}$.
4. $\mathcal{N} \subseteq \mathcal{B}$.
5. Player I *does not have a winning strategy for* BM($\mathcal{N}$).
6. Player II *has a winning strategy for* BM($\mathcal{N}$).

Next we show a measure-theoretic analogue of Theorem 3.2 under an additional assumption.

Theorem 3.4. *Suppose* $\mu(\mathcal{B}) < 1$. *Then the following conditions are equivalent.*

1. $\mathrm{HTP}(\mathbb{Q}) >_{\mathrm{T}} \emptyset$, *i.e.,* HTP($\mathbb{Q}$) *is undecidable.*
2. $\mu(\mathcal{N}) = 1$.
3. $\mu(\mathcal{N}) > \mu(\mathcal{B})$.
4. $\mathcal{N} \cap \overline{\mathcal{B}} \neq \emptyset$.

This theorem can also be viewed as a zero-one law if, in addition, $\mu(\mathcal{B}) = 0$.

Proof. By a result due to Sacks, the Turing upper cone over a set $A \in 2^\omega$ has measure 0 unless A is computable (see §10 THEOREM 1 in [Sac64]). Therefore the set of the sets in which HTP($\mathbb{Q}$) is not computable has measure 1. The rest of the proof is almost same as in Theorem 3.2. □

However, the assumption in Theorem 3.4 contradicts diophantineness of $\mathbb{Z}$ in $\mathbb{Q}$ because of the following Miller's result.

Theorem 3.5 (Miller, Theorem 6.7 in [Mil19])**.** *If* $\mu(\mathcal{B}) < 1$, *then* $\mathbb{Z}$ *is not a diophantine set in* $\mathbb{Q}$.

So, Theorem 3.4 does not hold if $\mathbb{Z}$ is diophantine in $\mathbb{Q}$, contrary to Mazur's conjecture.

An important point for our main theorems is the inequality $(*)$. It can also be satisfied by HTP-complete sets, which is introduced by Miller in [Mil19]. A set $W \in 2^{\mathbb{P}}$ is called *HTP-complete* if $W' \leq_1 \mathrm{HTP}(R_W)$. Note that every HTP-complete set is also HTP-nontrivial. However, it is known that there are only few HTP-complete sets in $2^{\mathbb{P}}$.

Theorem 3.6 (Theorem 3.2 in [Mil19], cf. Corollary 3.3 in [KM19])**.** *The set of all HTP-complete subsets of* $\mathbb{P}$ *is meager within the power set of* $\mathbb{P}$ *and has Lebesgue measure* 0.

Therefore HTP-complete sets is not suitable for Theorem 3.2. Moreover, at present, we cannot rule out the possibility that $\mathrm{HTP}(\mathbb{Q})$ is undecidable but there is no set $W \in 2^{\mathbb{P}}$ such that W is both HTP-generic and HTP-complete.

3.2 Constructing HTP-generic set via Banach-Mazur game

The following proposition generalizes Proposition 3.1 in [EMPS17] in terms of Banach-Mazur game.

Proposition 3.7. *For a string $\sigma \in 2^{<\mathbb{P}}$ and a real number $r \in [0, 1)$ which is computably approximable from above, there exists a set $W \in 2^{\mathbb{P}}$ such that*

1. $\sigma \prec W$.
2. $W \leq_{\mathrm{T}} \mathrm{HTP}(\mathbb{Q})$.
3. *W is HTP-generic.*
4. $\mathrm{HTP}(R_W) \leq_{\mathrm{T}} \mathrm{HTP}(\mathbb{Q})$.
5. *The lower density of W is r, i.e.,*

$$\liminf_{n\to\infty} \frac{|W \cap \{p_0, p_1, \dots, p_{n-1}\}|}{n} = r.$$

In particular, W is a co-infinite set with $\mathrm{HTP}(R_W) \equiv_{\mathrm{T}} \mathrm{HTP}(\mathbb{Q})$.

Proof. Fix a computable enumeration $(f_s)_{s\in\omega}$ of the polynomials in $\mathbb{Z}[x_1, x_2, \dots]$ and a computable decreasing sequence $(q_s)_{s\in\omega}$ of rational numbers such that $\lim_{s\to\infty} q_s = r$. Now each player follows the following strategies.

- Define $\sigma_0 = \sigma$. For a given σ_{2s+1}, Player I checks whether $f_s \in \mathrm{HTP}(R_{\sigma_{2s+1}{}^\frown 1^\infty})$. If so, take a solution $\vec{x} \in (R_{\sigma_{2s+1}{}^\frown 1^\infty})^n$ of $f_s = 0$. Let $m \in \omega$ be the minimum number such that $\vec{x} \in (R_{\sigma_{2s+1}{}^\frown 1^m 0^\infty})^n$. Then Player I chooses $\sigma_{2s+2} = \sigma_{2s+1}{}^\frown 1^{m \dot{-} |\sigma_{2s+1}|}$ so that $\mathcal{U}_{\sigma_{2s+2}} \cap \mathcal{B}(f_s) = \emptyset$. If $f_s \notin \mathrm{HTP}(R_{\sigma_{2s+1}{}^\frown 1^\infty})$, define $\sigma_{2s+2} = \sigma_{2s+1}{}^\frown 1$.
- For a given σ_{2s}, let $m \in \omega$ be the minimum number such that

$$\frac{|\sigma_{2s}{}^\frown 0^m|_1}{|\sigma_{2s}{}^\frown 0^m|} \leq q_s + 2^{-s}$$

  where $|\sigma_{2s}{}^\frown 0^m|_1$ denotes the number of 1's occurring in $\sigma_{2s}{}^\frown 0^m$. Then Player II chooses $\sigma_{2s+1} = \sigma_{2s}{}^\frown 0^m$.

The resulting set $W = \bigcup_{n\in\omega} \sigma_n \succ \sigma$ is computable in $\mathrm{HTP}(\mathbb{Q})$ since the both strategies are computable in $\mathrm{HTP}(\mathbb{Q})$. The HTP-genericity of W follows from the strategy for Player I. Then it follows from Proposition 2.5 that $\mathrm{HTP}(R_W) \leq_{\mathrm{T}} W \oplus \mathrm{HTP}(\mathbb{Q}) \leq_{\mathrm{T}} \mathrm{HTP}(\mathbb{Q})$. The lower density of W is r because of the strategy for Player II. □

Note that the strategy for Player II in the proof is a winning strategy for $\mathrm{BM}(\overline{\mathcal{B}})$.

3.3 Partial results and Questions

Theorem 3.8. *The set $\mathcal{N}_{\mathrm{m}}$ of the m-nontrivial sets is comeager in $2^{\mathbb{P}}$.*

Proof. For each computable function $h \colon \mathbb{Z}[x_1, x_2, \dots] \to \omega$, put $\mathcal{X}_h := \{\, W \mid \mathrm{HTP}(R_W) \leq_{\mathrm{m}} W \text{ via } p_{(-)} \circ h \,\}$. Note that $\mathcal{N}_{\mathrm{m}} = \bigcap_h \overline{\mathcal{X}_h}$. Since $\overline{\mathcal{B}}$ is comeager in $2^{\mathbb{P}}$, it suffices to show that each $\mathcal{X}_h \cap \overline{\mathcal{B}}$ is closed and nowhere dense in $\overline{\mathcal{B}}$.

First we show that $\overline{\mathcal{X}_h} \cap \overline{\mathcal{B}}$ is an open set in $\overline{\mathcal{B}}$. Let $W \in \overline{\mathcal{X}_h} \cap \overline{\mathcal{B}}$. Then there exists a polynomial f which satisfies one of the following conditions.

1. $f \in \mathrm{HTP}(R_W) \wedge p_{h(f)} \notin W$.
2. $f \notin \mathrm{HTP}(R_W) \wedge p_{h(f)} \in W$.

If (1) holds, then there exists some $n \in \omega$ such that $f \in \mathrm{HTP}(R_{(W\upharpoonright n)^\frown 0^\infty})$. Choose a sufficiently long initial segment $\sigma \prec W$ so that $|\sigma| > \max\{n, h(f)\}$. Then any HTP-generic set $V \succ \sigma$ satisfies $f \in \mathrm{HTP}(R_V) \wedge p_{h(f)} \notin V$. Suppose (2) is true. Then $W \in \mathcal{C}(f)$ since W is HTP-generic. Hence there exists some $n \in \omega$ such that $f \notin \mathrm{HTP}(R_U)$ for any $U \succ W\upharpoonright n$. Then similarly an initial segment $\sigma \prec W$ with $|\sigma| > \max\{n, h(f)\}$ ensures that any HTP-generic set $V \succ \sigma$ satisfies $f \notin \mathrm{HTP}(R_V) \wedge p_{h(f)} \in V$.

Next we show that $\mathcal{X}_h \cap \overline{\mathcal{B}}$ has no interior point in $\overline{\mathcal{B}}$. Let $W \in \mathcal{X}_h \cap \overline{\mathcal{B}}$ and $\sigma \prec W$. Put $n = |\sigma|$ and $f = (p_n x - 1)^2 + (p_{n+1} y - 1)^2$. Then $\mathcal{A}(f) = \{\, U \in 2^{\mathbb{P}} \mid p_n \in U \wedge p_{n+1} \in U \,\}$. Define

$$\tau = \begin{cases} \sigma^\frown 11 & \text{if } h(f) < n \wedge \sigma(h(f)) = 0, \\ \sigma^\frown 00 & \text{if } h(f) < n \wedge \sigma(h(f)) = 1, \\ \sigma^\frown 01^\frown 1^{h(f) \dot{-} (n+1)} & \text{if } h(f) \geq n \wedge h(f) \neq n, \\ \sigma^\frown 10^\frown 1^{h(f) \dot{-} (n+1)} & \text{if } h(f) \geq n \wedge h(f) \neq n+1. \end{cases}$$

By Proposition 3.7, there exists an HTP-generic set $V \succ \tau$, which satisfies $V \in \overline{\mathcal{X}_h} \cap \overline{\mathcal{B}}$. $\square$

For each total Turing functional Φ, put $\mathcal{X}_\Phi := \{\, W \in 2^{\mathbb{P}} \mid \mathrm{HTP}(R_W) = \Phi^W \,\}$. Then one can prove that $\mathcal{X}_\Phi \cap \overline{\mathcal{B}}$ is closed in $\overline{\mathcal{B}}$ in a similar manner to Theorem 3.8. However, it is not clear that whether $\mathcal{X}_\Phi \cap \overline{\mathcal{B}}$ has an internal point in $\overline{\mathcal{B}}$.

Theorem 3.9. *The set $\mathcal{N}_1$ of the 1-nontrivial sets has measure* 1.

Proof. Fix a computable enumeration $(f_s)_{s\in\omega}$ of the polynomials in $\mathbb{Z}[x_1, x_2, \dots]$. Let $h \colon \mathbb{Z}[x_1, x_2, \dots] \to \mathbb{P}$ be a computable injection. It suffices show that $\mathcal{X}_h := \{\, W \in 2^{\mathbb{P}} \mid \mathrm{HTP}(R_W) \leq_1 W \text{ via } h \,\}$ has measure 0. We inductively construct two functions $k, l \colon \omega \to \omega$ such that for all $s \in \omega$,

1. $\mathcal{X}_h \subseteq \{\, W \in 2^{\mathbb{P}} \mid p_{k(s)} \in W \iff h(f_{l(s)}) \in W \,\}$ and
2. $k(s) < h(f_{l(s)}) < k(s+1)$.

Stage $s = 0$. Put $k(0) = 0$.
Stage $s+1$. Since h is injective, there exists some $m \in \mathbb{Z}$ such that $h(m(p_{k(s)}x - 1)) > k(s)$. Let $l(s) \in \omega$ be the unique number such that $f_{l(s)} = m(p_{k(s)}x - 1)$ and put $k(s+1) = h(f_{l(s)}) + 1$.

It follows from (2) that

$$\begin{aligned} \mu(\mathcal{X}_h) &\leq \mu\left(\bigcap_{s\in\omega} \{\, W \in 2^{\mathbb{P}} \mid p_{k(s)} \in W \iff h(f_{l(s)}) \in W \,\} \right) \\ &= \lim_{s\to\infty} \mu\left(\bigcap_{t<s} \{\, W \in 2^{\mathbb{P}} \mid p_{k(t)} \in W \iff h(f_{l(t)}) \in W \,\} \right) \\ &= \lim_{s\to\infty} 2^{-s} = 0. \end{aligned}$$

$\square$

Question 3.10. How about the Baire category of $\mathcal{N}_{\mathrm{tt}}$ and $\mathcal{N}_{\mathrm{bT}}$?

Question 3.11. How about the measure of $\mathcal{N}_{\mathrm{m}}$, $\mathcal{N}_{\mathrm{tt}}$ and $\mathcal{N}_{\mathrm{bT}}$?

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