Groups whose word problems are accepted by abelian G-automata

Takao Yuyama

Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan
yuyama@kurims.kyoto-u.ac.jp

Abstract. Elder, Kambites, and Ostheimer showed that if a finitely generated group H has word problem accepted by a G-automaton for an abelian group G, then H has an abelian subgroup of finite index. Their proof is, however, non-constructive in the sense that it is by contradiction: they proved that H must have a finite index abelian subgroup without constructing any finite index abelian subgroup of H. In addition, a part of their proof is in terms of geometric group theory, which makes it hard to read without knowledge of the field.
We give a new, elementary, and in some sense more constructive proof of the theorem, in which we construct, from the abelian G-automaton accepting the word problem of H, a group homomorphism from a subgroup of G onto a finite index subgroup of H. Our method is purely combinatorial and contains no geometric arguments.

Keywords: word problem · G-automaton · abelian group.

1 Introduction

For a group G, a *G-automaton* is a variant of usual finite state automata, which is augmented with a memory register that stores an element of G. During the computation of a G-automaton, the content of the register may be updated by multiplying on the right by an element of G, but cannot be seen. Such an automaton first initializes the register with the identity element 1_G of G, and the automaton accepts an input word if, by reading this word, it can reach a terminal state, in which the register content is 1_G. (For the precise definition, see Section 2.4.) For a positive integer n, $\mathbb{Z}^n$-automata are the same as *blind n-counter automata*, which were defined and studied by Greibach [14,15]. Note that the notion of G-automata is discovered repeatedly by several different authors. The name "G-automaton" is due to Kambites [22]. (In fact, they introduced the notion of M-automata for any monoid M.) Render–Kambites [31] uses *G-valence automata* and Dassow–Mitrana [8] and Mitrana–Stiebe [26] use *extended finite automata* (EFA) over G instead of G-automata.

For a finitely generated group H, the *word problem* of H, with respect to a fixed finite generating set of H, is the set of words over the generating set representing the identity element of H (see Section 2.2 for the precise definitions). For several language classes, the class of finitely generated groups whose word problem is in the class has been determined [1,27,2,17,10,19,28], and many attempts are made for other language classes [3,4,20,12,24,21,25,13,30]. One of the most remarkable theorems about word problems is the well-known result due to Muller and Schupp [27], which states that, with the theorem by Dunwoody [9], a group has a context-free word problem if and only if it is virtually free, i.e., has a free subgroup of finite index. These theorems suggest deep connections between group theory and formal language theory.

Involving both G-automata and word problems, the following broad question was posed implicitly by Elston and Ostheimer [11] and explicitly by Kambites [22].

Question 1. For a given group G, is there any connection between the structural property of G and of the collection of groups whose word problems are accepted by *non-deterministic* G-automata?

Note that by G-automata, we always mean non-deterministic G-automata in this paper. As for *deterministic* G-automata, the following theorem is known.

Theorem 1 (Kambites [22, Theorem 1], 2006). *Let G and H be groups with H finitely generated. Then the word problem of H is accepted by a* deterministic *G-automaton if and only if H has a finite index subgroup which embeds in G.*

For non-deterministic G-automata, several results are known for specific types of groups. For a free group F of rank ≥ 2, it is known that a language is accepted by an F-automaton if and only if it is context-free (essentially by [5, Proposition 2], see also [7, Corollary 4.5] and [23, Theorem 7]). Combining with the Muller–Schupp theorem, the class of groups whose word problems are accepted by F-automata is the class of virtually free groups. The class of groups whose word problems are accepted by $(F \times F)$-automata is exactly the class of recursively presentable groups [7, Corollary 3.5][23, Theorem 8][26, Theorem 10].

For the case where G is (virtually) abelian, the following result was shown by Elder, Kambites, and Ostheimer. Recall that a group G is called virtually abelian if it has an abelian subgroup of finite index.

Theorem 2 (Elder, Kambites, and Ostheimer [10], 2008).

(1) *Let H be a finitely generated group and n be a positive integer. Then the word problem of H is accepted by a $\mathbb{Z}^n$-automaton if and only if H is virtually free abelian of rank at most n* [10, Theorem 1].
(2) *Let G be a virtually abelian group and H be a finitely generated group. Then the word problem of H is accepted by a G-automaton if and only if H has a finite index subgroup which embeds in G* [10, Theorem 4].

However, their proof is non-constructive in the sense that it is by contradiction: they proved that H must have a finite index abelian subgroup without constructing any finite index abelian subgroup of H. In addition, their proof depends on a deep theorem in geometric group theory due to Gromov [16], which states that every finitely generated group with polynomial growth function is virtually nilpotent.

The proof of Theorem 2 in [10] proceeds as follows. Let H be a finitely generated group whose word problem is accepted by a $\mathbb{Z}^n$-automaton. First, some techniques to compute several bounds for linear maps and semilinear sets are developed. Then a map from H to $\mathbb{Z}^n$ with some geometric conditions is constructed to prove that H has polynomial growth function. By Gromov's theorem, H is virtually nilpotent. Finally, it is proved that H is virtually abelian, using some theorems about nilpotent groups and semilinear sets. Theorem 2 (2) is deducible from Theorem 2 (1). Because of the non-constructivity of the proof, the embedding in Theorem 2 (2) is obtained only *a posteriori* and hence has nothing to do with the G-automaton.

To our knowledge, there are almost no attempts so far to obtain explicit algebraic connections between G and H, where H is a finitely generated group with word problem accepted by a G-automaton. The only exception is the result due to Holt, Owens, and Thomas [19, Theorem 4.2], where they gave a somewhat combinatorial proof to a special case of Theorem 2 (1), for the case where $n = 1$. (In fact, their theorem is slightly stronger than Theorem 2 (1) for $n = 1$ because it is for *non-blind* one-counter automata. See also [10, Section 7].) However, their proof also involves growth functions.

In this paper, we give an elementary, purely combinatorial proof of the following our main theorem, which is equivalent to Theorem 2 (see Section 3).

Theorem 3. *Let G be an abelian group and H be a finitely generated group. Suppose that the word problem of H is accepted by a G-automaton. Then there exists a group homomorphism from a subgroup of G onto a finite index subgroup of H.*

Our proof of Theorem 3 proceeds as follows. Suppose that the word problem of a finitely generated group H is accepted by a G-automaton A, where G is an abelian group. First, we prove that there exist only finitely many *minimal accepting paths* in A. Next, for each vertex p of A and each minimal accepting path μ in A, we define a set $M(\mu, p)$ of closed paths that is *pumpable* in μ and starts from p, and prove that each $M(\mu, p)$ forms a monoid with respect to concatenation. Then, we show that each monoid $M(\mu, p)$ induces a group homomorphism $f_{\mu,p}$ from a subgroup $G(\mu, p)$ of G onto a subgroup $H(\mu, p)$ of H. Finally, we show that at least one of the $H(\mu, p)$'s has finite index in H.

In addition to this introduction, this paper comprises four sections. Section 2 provides necessary preliminaries, notations, and conventions. In Section 3, we reduce Theorem 2 to Theorem 3 and vice versa. Section 4 is devoted to the proof of Theorem 3. Section 5 concludes the paper.

2 Preliminaries

2.1 Words, subwords, and scattered subwords

For a set Σ, we write Σ^* for the free monoid generated by Σ, i.e., the set of *words* over Σ. For a word $u = a_1 a_2 \cdots a_n \in \Sigma^*$ ($n \geq 0, a_i \in \Sigma$), the number n is called the *length* of u, which is denoted by $|u|$. For two words $u, v \in \Sigma^*$, the *concatenation* of u and v are denoted by $u \cdot v$, or simply uv. The identity element of Σ^* is the *empty word*, denoted by ε, which is the unique word of length zero. For an integer $n \geq 0$, the n-fold concatenation of a word $u \in \Sigma^*$ is denoted by u^n. For an integer $n > 0$, we write $\Sigma^{<n}$ for the set of words of length less than n.

A word $u \in \Sigma^*$ is a *subword* of a word $v \in \Sigma^*$, denoted by $u \sqsubseteq v$, if there exist two words $u_1, u_2 \in \Sigma^*$ such that $u_1 u u_2 = v$. A word $u \in \Sigma^*$ is a *scattered subword* of a word $v \in \Sigma^*$, denoted by $u \sqsubseteq_{\mathrm{sc}} v$, if there exist two finite sequences of words $u_1, u_2, \ldots, u_n \in \Sigma^*$ ($n \geq 0$) and $v_0, v_1, \ldots, v_n \in \Sigma^*$ such that $u = u_1 u_2 \cdots u_n$ and $v = v_0 u_1 v_1 u_2 v_2 \cdots u_n v_n$. That is, v is obtained by inserting some words in u. Note that the two binary relations $\sqsubseteq$ and $\sqsubseteq_{\mathrm{sc}}$ are both partial orderings on Σ^*.

2.2 Word problem for groups

Let H be a finitely generated group. A *choice of generators* for H is a surjective monoid homomorphism ρ from the free monoid Σ^*, on a finite alphabet Σ, onto H. The *word problem* of H with respect to ρ, denoted by $\mathrm{WP}_\rho(H)$, is the set of words in Σ^* mapped to the identity element 1_H of H via ρ, i.e., $\mathrm{WP}_\rho(H) = \rho^{-1}(1_H)$.

Although the word problem $\mathrm{WP}_\rho(H)$ depends on the choice of generators ρ, this does not cause problems, at least for our purpose:

Proposition 1 (e.g., [20, Lemma 1]). *Let $\mathcal{C}$ be a class of languages closed under inverse homomorphisms and let H be a finitely generated group. Then* $\mathrm{WP}_\rho(H) \in \mathcal{C}$ *for* some *choice of generators ρ if and only if* $\mathrm{WP}_\rho(H) \in \mathcal{C}$ *for* every *choice of generators ρ.* □

Therefore we usually say "*the* word problem of H" rather than "*a* word problem of H."

2.3 Graphs and paths

A *graph* is a 4-tuple $(V, E, \mathsf{s}, \mathsf{t})$, where V is the set of vertices, E is the set of (directed) edges, $\mathsf{s}\colon E \to V$ and $\mathsf{t}\colon E \to V$ are functions assigning to every edge $e \in E$ the *source* $\mathsf{s}(e) \in V$ and the *target* $\mathsf{t}(e) \in V$, respectively. A graph is *finite* if it has only finitely many vertices and edges.

A *path* (of *length* n) in a graph $\Gamma = (V, E, \mathsf{s}, \mathsf{t})$ is a word $e_1 e_2 \cdots e_n \in E^*$ ($n \geq 0$) of edges $e_i \in E$ such that $\mathsf{t}(e_i) = \mathsf{s}(e_{i+1})$ for $i = 1, 2, \ldots, n-1$. We usually use Greek letters for paths in a graph. For a non-empty path $\omega = e_1 e_2 \cdots e_n \in E^*$, the source and the target of ω are defined as $\mathsf{s}(\omega) = \mathsf{s}(e_1)$ and $\mathsf{t}(\omega) = \mathsf{t}(e_n)$, respectively. If $\omega = e_1 e_2 \cdots e_n$ and $\omega' = e'_1 e'_2 \cdots e'_k$ are non-empty paths such that $\mathsf{t}(\omega) = \mathsf{s}(\omega')$, or at least one of ω and ω' is empty, then the concatenation of ω and ω', denoted by $\omega \cdot \omega'$ or $\omega\omega'$, is the path $e_1 e_2 \cdots e_n e'_1 e'_2 \cdots e'_k$ of length $n + k$, i.e., the concatenation as words. A path ω in Γ is *closed* if $\mathsf{s}(\omega) = \mathsf{t}(\omega)$, or $\omega = \varepsilon$. For a closed path σ and an integer $n \geq 0$, we write σ^n for the n-fold concatenation of σ.

For a graph $\Gamma = (V, E, \mathsf{s}, \mathsf{t})$, an *edge-labeling function* is a function ℓ from E to a set M. If M is a monoid and $\omega = e_1 e_2 \cdots e_n \in E^*$ is a path in Γ, then the label of ω is defined as $\ell(\omega) = \ell(e_1)\ell(e_2) \cdots \ell(e_n)$ via the multiplication of M.

2.4 G-automata

For a group G, a (non-deterministic) G-*automaton* over a finite alphabet Σ is defined as a 5-tuple $(\Gamma, \ell_G, \ell_\Sigma, p_{\mathrm{init}}, p_{\mathrm{ter}})$, where $\Gamma = (V, E, \mathsf{s}, \mathsf{t})$ is a finite graph, $\ell_G \colon E \to G$ and $\ell_\Sigma \colon E \to \Sigma^*$ are edge-labeling functions, $p_{\mathrm{init}} \in V$ is the *initial vertex*, and $p_{\mathrm{ter}} \in V$ is the *terminal vertex*. For simplicity, we assume that $\ell_\Sigma(e) \in \Sigma \cup \{\varepsilon\}$ for each $e \in E$. (Note that this assumption does not decrease the accepting power of G-automata. Indeed, if necessary, one can subdivide an edge e with labels $\ell_\Sigma(e) = uv, \ell_G(e) = g$ into two new edges e_1, e_2 with labels $\ell_\Sigma(e_1) = u, \ell_G(e_1) = g$ and $\ell_\Sigma(e_2) = v, \ell_G(e_2) = 1_G$.) An *accepting path* in a G-automaton $A = (\Gamma, \ell_G, \ell_\Sigma, p_{\mathrm{init}}, p_{\mathrm{ter}})$ is a path α in Γ such that $\mathsf{s}(\alpha) = p_{\mathrm{init}}$, $\mathsf{t}(\alpha) = p_{\mathrm{ter}}$, and $\ell_G(\alpha) = 1_G$ (we consider that the empty path $\varepsilon \in E^*$ is accepting if and only if $p_{\mathrm{init}} = p_{\mathrm{ter}}$). We say that a path ω in Γ is *promising* if ω is a subword of some accepting path in A, i.e., there exist two paths $\omega_1, \omega_2 \in E^*$ such that the concatenation $\omega_1 \omega \omega_2 \in E^*$ is an accepting path in A. The *language accepted* by a G-automaton A, denoted by $L(A)$, is the set of all words $u \in \Sigma^*$ such that u is the label of some accepting path in A, i.e.,

$$L(A) = \{\, \ell_\Sigma(\alpha) \in \Sigma^* \mid \alpha \text{ is an accepting path in } A \,\}.$$

We say that a G-automaton A is *abelian* if G is an abelian group.

The class of languages accepted by G-automata satisfies the assumption of Proposition 1:

Proposition 2 (e.g., [23, Proposition 2]). *For a group G, the class of languages accepted by G-automata is closed under inverse homomorphisms.* □

Therefore one can speak of a group H whose word problem is accepted by a G-automaton without any reference to generating set for H.

3 Equivalence of Theorem 3 and Theorem 2

Proposition 3. *Theorem 3 implies Theorem 2.*

Proof. Since Theorem 2 (2) is deducible from Theorem 2 (1) [10, Section 6], it suffices to show Theorem 2 (1). If H is a finitely generated group and $\mathbb{Z}^m$ is a finite index subgroup of H for some $m \leq n$, then one can easily construct a $\mathbb{Z}^n$-automaton that accepts the word problem of H (see e.g., [11, Theorem 7]). Conversely, suppose that the word problem of H is accepted by a $\mathbb{Z}^n$-automaton. By Theorem 3, there exists a group homomorphism f from a subgroup G_0 of $\mathbb{Z}^n$ onto a finite index subgroup H_0 of H. In general, a subgroup S of a free abelian group F is also free abelian, and the rank of S does not exceed that of F (see e.g., [32, 4.2.3]). Thus $G_0 \cong \mathbb{Z}^m$ for some $m \leq n$. Since H_0 is a homomorphic image of $\mathbb{Z}^m$, H_0 is an abelian group generated by at most m elements. Hence, by the fundamental theorem of finitely generated abelian groups (see e.g., [32, 4.2.10]), H_0 has a finite index subgroup isomorphic to $\mathbb{Z}^k$ for some $k \leq m$. Thus H has a finite index subgroup isomorphic to $\mathbb{Z}^k$. □

Proposition 4. *Theorem 2 implies Theorem 3.*

Proof. Suppose that the word problem of a finitely generated group H is accepted by a G-automaton A, where G is an abelian group. By Theorem 2 (2), there exist a finite index subgroup H_0 of H and an embedding $f\colon H_0 \to G$. Since f is injective, the homomorphism $f^{-1}\colon f(H_0) \to H_0$ is the desired one.

4 Proof of Theorem 3

Throughout this section, we fix an abelian group G, a finitely generated group H, a choice of generators $\rho\colon \Sigma^* \to H$, and an abelian G-automaton $A = (\Gamma = (V, E, \mathsf{s}, \mathsf{t}), \ell_G, \ell_\Sigma, p_{\text{init}}, p_{\text{ter}})$ such that $\mathrm{WP}_\rho(H) = L(A)$. We write the group operation of G additively and write 0_G for the identity element of G.

4.1 Minimal accepting paths

The minimal accepting paths, defined below, play a central role in this paper.

Definition 1. *An accepting path α in A is* minimal *if it is minimal with respect to the scattered subword relation $\sqsubseteq_{\text{sc}}$ on E^* among the accepting paths. An accepting path α in A* dominates *a minimal accepting path μ in A if $\mu \sqsubseteq_{\text{sc}} \alpha$.*

A similar notion of minimal accepting paths can be found in [6, Section 4].

Proposition 5 (Higman's lemma [18, Theorem 4.4]). *For any finite alphabet Σ, the scattered subword relation $\sqsubseteq_{\text{sc}}$ on Σ^* is a* well-quasi-order*, i.e., for any infinite sequence $u_1, u_2, \ldots \in \Sigma^*$, there exist some $i < j$ such that $u_i \sqsubseteq_{\text{sc}} u_j$.* □

Corollary 1. *There are only finitely many minimal accepting paths in A, and every accepting path on A dominates some minimal accepting path in A.*

Proof. Suppose the contrary that there are infinitely many distinct minimal accepting paths $\mu_1, \mu_2, \ldots \in E^*$. Then we have $\mu_i \not\sqsubseteq_{\text{sc}} \mu_j$ for any $i < j$ because of the minimality of μ_j, a contradiction. The second half of the lemma is also true since a well-quasi-ordered set admits no infinite descending sequence. □

Note that if $p_{\text{init}} = p_{\text{ter}}$ then the only minimal accepting path is the empty path ε.

4.2 Pumpable paths and the monoids $M(\mu, p)$

Definition 2. *Let $\mu = e_1 e_2 \cdots e_n \in E^*$ ($e_i \in E$) be a minimal accepting path in A. A closed path $\sigma \in E^*$ in Γ is* pumpable in μ *if there exists an accepting path α in A dominating μ such that $\alpha = \alpha_0 e_1 \alpha_1 e_2 \cdots e_n \alpha_n$ for some paths $\alpha_0, \alpha_1, \ldots, \alpha_n \in E^*$ in Γ and $\sigma \sqsubseteq \alpha_j$ for some $j \in \{0, 1, \ldots, n\}$.*

Remark 1.

(1) In Definition 2, each α_i inserted to μ is a closed path in Γ. In addition, $\ell_G(\alpha_0) + \ell_G(\alpha_1) + \cdots + \ell_G(\alpha_n) = 0_G$ since G is abelian and $\ell_G(\alpha) = \ell_G(\mu) = 0_G$.
(2) Every closed path σ pumpable in a minimal accepting path μ is promising since σ is a subword (not a scattered subword) of an accepting path α dominating μ.

Definition 3. *For a minimal accepting path μ in A and a vertex $p \in V$, define*

$$M(\mu, p) = \left\{ \sigma \in E^* \;\middle|\; \begin{array}{c} \sigma \text{ is a closed path in } \Gamma \text{ pumpable in } \mu \\ \text{such that } \mathsf{s}(\sigma) = p, \text{ or } \sigma = \varepsilon \end{array} \right\}.$$

Note that there are only finitely many $M(\mu, p)$'s by Corollary 1.

Lemma 1. *Each $M(\mu, p)$ is a monoid with respect to the concatenation operation, i.e., $\sigma_1, \sigma_2 \in M(\mu, p)$ implies $\sigma_1 \sigma_2 \in M(\mu, p)$.*

Proof. Since both σ_1 and σ_2 are pumpable in $\mu = e_1 e_2 \cdots e_n \in E^*$ ($e_i \in E$), there exist two accepting paths $\alpha = \alpha_0 e_1 \alpha_1 e_2 \cdots e_n \alpha_n$ ($\alpha_i \in E^*$) and $\beta = \beta_0 e_1 \beta_1 e_2 \cdots e_n \beta_n$ ($\beta_i \in E^*$) such that $\sigma_1 \sqsubseteq \alpha_i$ and $\sigma_2 \sqsubseteq \beta_j$ for some $i, j \in \{0, 1, \ldots, n\}$. Then we have $\alpha_i = \alpha_i' \sigma_1 \alpha_i''$ for some $\alpha_i', \alpha_i'' \in E^*$ and $\beta_j = \beta_j' \sigma_2 \beta_j''$ for some $\beta_j', \beta_j'' \in E^*$. We may assume that $i \leq j$. Since G is abelian, the merged path $\gamma = (\alpha_0 \beta_0) e_1 (\alpha_1 \beta_1) e_2 \cdots e_n (\alpha_n \beta_n)$ and its permutation

$$\gamma' = (\alpha_0 \beta_0) e_1 (\alpha_1 \beta_1) e_2 \cdots e_i (\alpha_i' \sigma_1 \sigma_2 \alpha_i'' \beta_i) e_{i+1} \cdots e_j (\alpha_j \beta_j' \beta_j'') e_{j+1} \cdots e_n (\alpha_n \beta_n) \tag{1}$$

are accepting paths in A by Remark 1 (1) (Figure 1). □

Lemma 2. *Let σ and τ be closed paths in Γ such that $\mathsf{s}(\tau) = p$ (or $\tau = \varepsilon$) and $\tau \sqsubseteq_{\mathrm{sc}} \sigma \in M(\mu, p)$. Then $\tau \in M(\mu, p)$.*

Proof. Suppose that $\tau = e_1' e_2' \cdots e_k'$ ($k \geq 0, e_i' \in E$) and $\sigma = \sigma_0 e_1' \sigma_1 e_2' \cdots e_k' \sigma_k$ ($\sigma_i \in E^*$). Note that each σ_i is a closed path in Γ. Since, by Lemma 1, σ^2 is pumpable in $\mu = e_1 e_2 \cdots e_n$ ($n \geq 0, e_i \in E$), there exists an accepting path $\alpha = \alpha_0 e_1 \alpha_1 e_2 \cdots e_n \alpha_n$ dominating μ such that $\sigma^2 \sqsubseteq_{\mathrm{sc}} \alpha_i$ for some $i \in \{0, 1, \ldots, n\}$. If $\alpha_i = \alpha_i' \sigma^2 \alpha_i''$, then the path

$$\alpha_0 e_1 \alpha_1 e_2 \cdots e_i (\alpha_i' \cdot \tau \cdot (\sigma_0^2 e_1' \sigma_1^2 e_2' \cdots e_k' \sigma_k^2) \cdot \alpha_i'') e_{i+1} \cdots e_n \alpha_n \tag{2}$$

is an accepting path in A by Remark 1 (1) (Figure 2). □

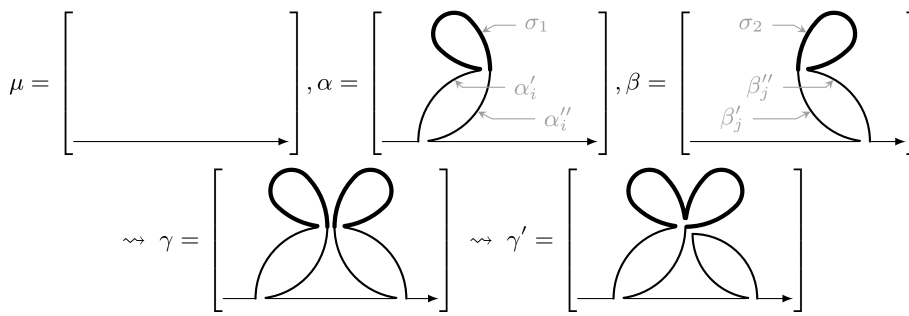

Fig. 1. Construction of the accepting path γ' in (1)

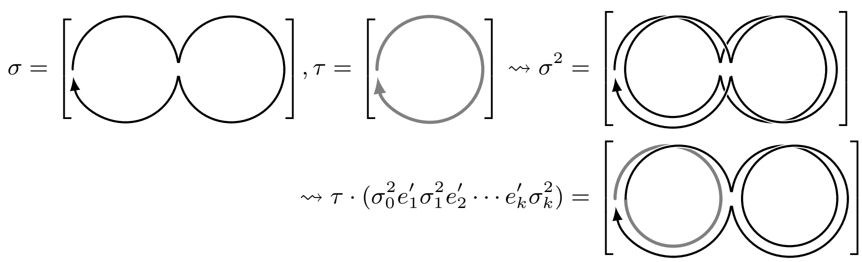

Fig. 2. Construction of the path $\tau \cdot (\sigma_0^2 e'_1 \sigma_1^2 e'_2 \cdots e'_k \sigma_k^2)$ in (2)

Lemma 3. *Let $\sigma \in M(\mu, p)$ and $\omega \sqsubseteq \sigma$ be a path. Then there exist two paths $\omega_1, \omega_2 \in E^{<|V|}$ such that $\omega_1 \omega \omega_2 \in M(\mu, p)$.*

Proof. Let $(\omega_1, \omega_2) \in E^* \times E^*$ be a pair of two paths such that $\omega_1 \omega \omega_2 \in M(\mu, p)$ and $\max\{|\omega_1|, |\omega_2|\}$ is minimum. Such a pair exists since $\omega \sqsubseteq \sigma \in M(\mu, p)$. Suppose the contrary that $\max\{|\omega_1|, |\omega_2|\} \geq |V|$, say $|\omega_1| \geq |V|$. By the pigeonhole principle, ω_1 must visit some vertex $p \in V$ at least twice. That is, there exist three paths α, β, γ such that $\omega_1 = \alpha\beta\gamma$ and β is a non-empty closed path. Now we have $\alpha\gamma\omega\omega_2 \sqsubseteq_{\mathrm{sc}} \omega_1\omega\omega_2 \in M(\mu, p)$, and Lemma 2 implies $\alpha\gamma\omega\omega_2 \in M(\mu, p)$, which contradicts the minimality of (ω_1, ω_2). □

4.3 Group homomorphisms $f_{\mu,p}$ from $G(\mu, p)$ onto $H(\mu, p)$

For each $M(\mu, p)$, Lemma 1 allows us to define a surjective monoid homomorphism $\varphi_{\mu,p} \colon M(\mu, p) \to \rho(\ell_\Sigma(M(\mu, p)))$ as the composition function $\rho \circ \ell_\Sigma$. Then each $\varphi_{\mu,p}$ induces a well-defined surjective monoid homomorphism $\bar{\varphi}_{\mu,p} \colon \ell_G(M(\mu, p)) \to \rho(\ell_\Sigma(M(\mu, p)))$ thanks to the following Lemma 4.

Lemma 4. *Let ω and ω' be paths in Γ such that $\mathsf{s}(\omega) = \mathsf{s}(\omega')$ and $\mathsf{t}(\omega) = \mathsf{t}(\omega')$, and suppose that ω is promising. Then $\ell_G(\omega) = \ell_G(\omega')$ implies $\rho(\ell_\Sigma(\omega)) = \rho(\ell_\Sigma(\omega'))$.*

Proof. Since ω is promising, there exist two paths ω_1, ω_2 in Γ such that $\omega_1\omega\omega_2$ is an accepting path in A. From the assumption $\ell_G(\omega) = \ell_G(\omega')$, we have $\ell_G(\omega_1\omega'\omega_2) = \ell_G(\omega_1\omega\omega_2) = 0_G$ and hence $\omega_1\omega'\omega_2$ is also an accepting path in A. That is, $\ell_\Sigma(\omega_1\omega\omega_2), \ell_\Sigma(\omega_1\omega'\omega_2) \in L(A) \subseteq \mathrm{WP}_\rho(H)$ and hence

$$\rho(\ell_\Sigma(\omega_1))\rho(\ell_\Sigma(\omega))\rho(\ell_\Sigma(\omega_2)) = 1_H = \rho(\ell_\Sigma(\omega_1))\rho(\ell_\Sigma(\omega'))\rho(\ell_\Sigma(\omega_2)).$$

Since H is cancellative, we have $\rho(\ell_\Sigma(\omega)) = \rho(\ell_\Sigma(\omega'))$. □

Let $G(\mu, p)$ (resp. $H(\mu, p)$) denote the subgroup of G generated by $\ell_G(M(\mu, p))$ (resp. the subgroup of H generated by $\rho(\ell_\Sigma(M(\mu, p))))$.

Lemma 5. *One can extend $\bar{\varphi}_{\mu,p} \colon \ell_G(M(\mu, p)) \to \rho(\ell_\Sigma(M(\mu, p)))$ to a unique surjective group homomorphism $f_{\mu,p} \colon G(\mu, p) \to H(\mu, p)$.*

Proof. Since G is an abelian group, every element $g \in G(\mu, p)$ can be written as $g = \ell_G(\sigma_1) - \ell_G(\sigma_2)$ for some $\sigma_1, \sigma_2 \in M(\mu, p)$. Defining $f_{\mu,p}(g) = \rho(\ell_\Sigma(\sigma_1)) - \rho(\ell_\Sigma(\sigma_2))$, one can easily check the well-definedness, the uniqueness, and the surjectivity. □

4.4 One of the $H(\mu, p)$'s has finite index in H

The remaining task is to prove that at least one of the $H(\mu, p)$'s has finite index in H. To do this, we use B. H. Neumann's lemma in the following form.

Proposition 6 (B. H. Neumann's lemma [29, (4.1) Lemma and (4.2)]). *Let H be a group, $H_1, H_2, \ldots, H_n$ be subgroups of H, and $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$ be elements of H. If $H = \bigcup_{i=1}^{n} a_i H_i b_i$, then at least one of the H_i's is of index at most n in H.* □

Lemma 6. *The following holds.*

$$H = \bigcup \left\{ h_1^{-1} H(\mu, p) h_2^{-1} \;\middle|\; \begin{array}{c} \mu \text{ is a minimal accepting path in } A, \\ p \in V, \text{ and } h_1, h_2 \in \rho(\Sigma^{<|V|}) \end{array} \right\}. \tag{3}$$

Proof. Let $h \in H$ and fix a word $v \in \Sigma^*$ such that $\rho(v) = h$. Since ρ is surjective, there exists a word $\bar{v} \in \Sigma^*$ such that $\rho(\bar{v}) = \rho(v)^{-1}$. Define

$$N = 1 + \max\{\, |\mu| \mid \mu \in E^* \text{ is a minimal accepting path in } A \,\},$$

and we have $N < \infty$ by Corollary 1. Since $(v\bar{v})^N \in \mathrm{WP}_\rho(H) \subseteq L(A)$, there exists an accepting path

$$\alpha = \omega_1\bar{\omega}_1\omega_2\bar{\omega}_2 \cdots \omega_N\bar{\omega}_N \tag{4}$$

in A such that $\ell_\Sigma(\omega_i) = v$ and $\ell_\Sigma(\bar{\omega}_i) = \bar{v}$ for $i = 1, 2, \ldots, N$. Let $\mu = e_1e_2 \cdots e_n$ ($e_i \in E$) be a minimal accepting path such that α dominates μ. Then we have another decomposition

$$\alpha = \alpha_0 e_1 \alpha_1 e_2 \cdots e_n \alpha_n \tag{5}$$

for some closed paths $\alpha_0, \alpha_1, \ldots, \alpha_n \in E^*$. Since $N > |\mu| = n$ and each e_i in the decomposition (5) is contained in at most one ω_i in the decomposition (4), at least one of the ω_i's is "disjoint" from all e_i's, i.e., there exist $i \in \{1, 2, \ldots, N\}$ and $j \in \{0, 1, \ldots, n\}$ such that $\omega_i \sqsubseteq \alpha_j$. Since α_j is a pumpable closed path in μ, we have $\alpha_j \in M(\mu, p)$, where $p = \mathsf{s}(\alpha_j)$. By Lemma 3, there exist $\alpha'_j, \alpha''_j \in E^{<|V|}$ such that $\alpha'_j \omega_i \alpha''_j \in M(\mu, p)$. Then we have $|\ell_\Sigma(\alpha'_j)|, |\ell_\Sigma(\alpha''_j)| < |V|$ and $\rho(\ell_\Sigma(\alpha'_j))\rho(\ell_\Sigma(\omega_i))\rho(\ell_\Sigma(\alpha''_j)) \in \rho(\ell_\Sigma(M(\mu, p))) \subseteq H(\mu, p)$, hence

$$h = \rho(v) = \rho(\ell_\Sigma(\omega_i)) \in \rho(\ell_\Sigma(\alpha'_j))^{-1} H(\mu, p) \rho(\ell_\Sigma(\alpha''_j))^{-1}.$$

Thus (3) holds. □

Proof of Theorem 3. Since G is an abelian group and $L(A) \subseteq \mathrm{WP}_\rho(H)$, we have surjective group homomorphisms $f_{\mu,p} \colon G(\mu, p) \to H(\mu, p)$ by Lemmata 4 and 5. Since $\mathrm{WP}_\rho(H) \subseteq L(A)$, the equation (3) holds by Lemma 6, and the right-hand side of (3) is a finite union of cosets of H by Corollary 1. Thus, by B. H. Neumann's lemma (Proposition 6), at least one of the $H(\mu, p)$'s is of index at most $|\{\, \mu \mid \mu \text{ is a minimal accepting path in } A \,\}| \times |V| \times |\Sigma^{<|V|}|^2$ in H. □

5 Conclusion

We gave a new, elementary, purely combinatorial proof of the theorem due to Elder, Kambites, and Ostheimer, which states that if a finitely generated group H has word problem accepted by an abelian G-automaton A, then H is virtually abelian. In contrast to their original geometric argument, we stuck to the combinatorial link—the abelian G-automaton A—between the two groups G and H, and we obtained explicit algebraic connections between them as finitely many group homomorphisms $f_{\mu,p} \colon G(\mu, p) \to H(\mu, p)$.

We leave the following question as our future work.

Question 2. Let G and H be (not necessarily abelian) groups with H finitely generated. Suppose that the word problem of H is accepted by a G-automaton. Can one combinatorially obtain a group homomorphism from a subgroup of G onto a finite index subgroup of H?

For example, how about the case where G is free or nilpotent? Our Theorem 3 is the very first step for approaching Question 2.

Acknowledgements The author would like to thank Ryoma Sin'ya (Akita University) for encouragement and helpful comments. They would also like to thank anonymous reviewers for many valuable comments. This work was supported by JSPS KAKENHI Grant Numbers JP20J23039 and JP20H05961.

References

1. Anīsīmov, A.V.: Certain algorithmic questions for groups and context-free languages. Kibernetika (Kiev) (2), 4–11 (1972)

2. Boone, W.W., Higman, G.: An algebraic characterization of groups with soluble word problem. J. Austral. Math. Soc. **18**, 41–53 (1974), collection of articles dedicated to the memory of Hanna Neumann, IX
3. Brough, T.: Groups with poly-context-free word problem. Groups Complex. Cryptol. **6**(1), 9–29 (2014), https://doi.org/10.1515/gcc-2014-0002
4. Ceccherini-Silberstein, T., Coornaert, M., Fiorenzi, F., Schupp, P.E., Touikan, N.W.M.: Multipass automata and group word problems. Theoret. Comput. Sci. **600**, 19–33 (2015), https://doi.org/10.1016/j.tcs.2015.06.054
5. Chomsky, N., Schützenberger, M.P.: The algebraic theory of context-free languages. In: Computer programming and formal systems, pp. 118–161. North-Holland, Amsterdam (1963)
6. Cleary, S., Elder, M., Ostheimer, G.: The word problem distinguishes counter languages (2006), https://arxiv.org/abs/math/0606415
7. Corson, J.M.: Extended finite automata and word problems. Internat. J. Algebra Comput. **15**(3), 455–466 (2005), https://doi.org/10.1142/S0218196705002360
8. Dassow, J., Mitrana, V.: Finite automata over free groups. Internat. J. Algebra Comput. **10**(6), 725–737 (2000), https://doi.org/10.1142/S0218196700000315
9. Dunwoody, M.J.: The accessibility of finitely presented groups. Invent. Math. **81**(3), 449–457 (1985), https://doi.org/10.1007/BF01388581
10. Elder, M., Kambites, M., Ostheimer, G.: On groups and counter automata. Internat. J. Algebra Comput. **18**(8), 1345–1364 (2008), https://doi.org/10.1142/S0218196708004901
11. Elston, G.Z., Ostheimer, G.: On groups whose word problem is solved by a counter automaton. Theoret. Comput. Sci. **320**(2-3), 175–185 (2004), https://doi.org/10.1016/j.tcs.2003.09.007
12. Gebhardt, K., Meunier, F., Salvati, S.: O_n is an n-MCFL. J. Comput. System Sci. **127**, 41–52 (2022), https://doi.org/10.1016/j.jcss.2022.02.003
13. Gilman, R.H., Kropholler, R.P., Schleimer, S.: Groups whose word problems are not semilinear. Groups Complex. Cryptol. **10**(2), 53–62 (2018), https://doi.org/10.1515/gcc-2018-0010
14. Greibach, S.A.: Remarks on the complexity of nondeterministic counter languages. Theoret. Comput. Sci. **1**(4), 269–288 (1975/76), https://doi.org/10.1016/0304-3975(76)90072-4
15. Greibach, S.A.: Remarks on blind and partially blind one-way multicounter machines. Theoret. Comput. Sci. **7**(3), 311–324 (1978), https://doi.org/10.1016/0304-3975(78)90020-8
16. Gromov, M.: Groups of polynomial growth and expanding maps. Inst. Hautes Études Sci. Publ. Math. (53), 53–73 (1981), http://www.numdam.org/item?id=PMIHES_1981__53__53_0
17. Higman, G.: Subgroups of finitely presented groups. Proc. Roy. Soc. London Ser. A **262**, 455–475 (1961), https://doi.org/10.1098/rspa.1961.0132
18. Higman, G.: Ordering by divisibility in abstract algebras. Proc. London Math. Soc. (3) **2**, 326–336 (1952), https://doi.org/10.1112/plms/s3-2.1.326
19. Holt, D.F., Owens, M.D., Thomas, R.M.: Groups and semigroups with a one-counter word problem. J. Aust. Math. Soc. **85**(2), 197–209 (2008), https://doi.org/10.1017/S1446788708000864
20. Holt, D.F., Rees, S., Röver, C.E., Thomas, R.M.: Groups with context-free co-word problem. J. London Math. Soc. (2) **71**(3), 643–657 (2005), https://doi.org/10.1112/S002461070500654X
21. Holt, D.F., Röver, C.E.: Groups with indexed co-word problem. Internat. J. Algebra Comput. **16**(5), 985–1014 (2006), https://doi.org/10.1142/S0218196706003359

22. Kambites, M.: Word problems recognisable by deterministic blind monoid automata. Theoret. Comput. Sci. **362**(1-3), 232–237 (2006), https://doi.org/10.1016/j.tcs.2006.06.026
23. Kambites, M.: Formal languages and groups as memory. Comm. Algebra **37**(1), 193–208 (2009), https://doi.org/10.1080/00927870802243580
24. Kanazawa, M., Salvati, S.: MIX is not a tree-adjoining language. In: Proceedings of the 50th Annual Meeting of the Association for Computational Linguistics (Volume 1: Long Papers). pp. 666–674. Association for Computational Linguistics, Jeju Island, Korea (Jul 2012), https://aclanthology.org/P12-1070
25. Kropholler, R.P., Spriano, D.: Closure properties in the class of multiple context-free groups. Groups Complex. Cryptol. **11**(1), 1–15 (2019), https://doi.org/10.1515/gcc-2019-2004
26. Mitrana, V., Stiebe, R.: Extended finite automata over groups. Discrete Appl. Math. **108**(3), 287–300 (2001), https://doi.org/10.1016/S0166-218X(00)00200-6
27. Muller, D.E., Schupp, P.E.: Groups, the theory of ends, and context-free languages. J. Comput. System Sci. **26**(3), 295–310 (1983), https://doi.org/10.1016/0022-0000(83)90003-X
28. Nesin, G.A.R., Thomas, R.M.: Groups whose word problem is a Petri net language. In: Descriptional complexity of formal systems, Lecture Notes in Comput. Sci., vol. 9118, pp. 243–255. Springer, Cham (2015), https://doi.org/10.1007/978-3-319-19225-3_21
29. Neumann, B.H.: Groups covered by permutable subsets. J. London Math. Soc. **29**, 236–248 (1954), https://doi.org/10.1112/jlms/s1-29.2.236
30. Nyberg-Brodda, C.F.: On the word problem for free products of semigroups and monoids. J. Algebra **622**, 721–741 (2023). https://doi.org/10.1016/j.jalgebra.2023.02.007, https://doi.org/10.1016/j.jalgebra.2023.02.007
31. Render, E., Kambites, M.: Polycyclic and bicyclic valence automata. In: Language and automata theory and applications, Lecture Notes in Comput. Sci., vol. 5196, pp. 464–475. Springer, Berlin (2008). https://doi.org/10.1007/978-3-540-88282-4_42, https://doi.org/10.1007/978-3-540-88282-4_42
32. Robinson, D.J.S.: A course in the theory of groups, Graduate Texts in Mathematics, vol. 80. Springer-Verlag, New York, second edn. (1996), https://doi.org/10.1007/978-1-4419-8594-1