Decidability of Presburger Arithmetic via Finite Automata

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An Ideal Machine

Suppose there were a machine like this:

Are there infinitely many prime numbers?

Yes.

Is there a nontrivial integer solution for $x^2 + y^2 = 3z^2$?

No.

Is Goldbach conjecture true?

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The machine eventually gives the correct answer to a question whether a given mathematical statement is true or not.

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Q. Can we create such a machine in the future?

A. No!

Moreover, there is no such a machine even for the statements about natural numbers.

Theorem (Gödel, 1931)

The first-order theory of the natural numbers is undecidable.

In other words, there are no algorithms for determining whether a given arithmetical sentence is true or not.

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The Aim of This Section

To clarify the meaning of the words

"first-order theory of the natural numbers"

and

"arithmetical sentence",

we introduce precise definitions from mathematical logic.

Available Symbols

Definition (vocabularies for arithmetic)

Define

$$\mathcal{L}_{\text{arith}} = \{\underline{0}, \underline{1}, +, \times, <\},$$
$$\mathcal{L}_{\text{Pres}} = \{\underline{0}, \underline{1}, +, <\}$$

where

- $\underline{0}$ and $\underline{1}$ are constant symbols,
- $+$ and $\times$ are binary function symbols,
- $<$ is a binary relation symbol.

Remark

Of course you may use $\mathcal{L}'_{\text{Pres}} = \{\underline{0}, \text{succ}, +, <\}$ instead of $\mathcal{L}_{\text{Pres}}$. However, $\mathcal{L}_{\text{Pres}}$ is more suitable for the algorithm we construct later.

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Terms

Terms express a natural number.

Let $\mathcal{L}$ be either $\mathcal{L}_{\text{arith}}$ or $\mathcal{L}_{\text{Pres}}$.

Definition ($\mathcal{L}$-terms)

1. All variable symbols $x, y, z, \ldots$ and constant symbols $\underline{0}$, $\underline{1}$ are $\mathcal{L}$-terms.
2. If t_1 and t_2 are two $\mathcal{L}$-terms, then
   - $(t_1 + t_2)$ and $(t_1 \times t_2)$ are $\mathcal{L}_{\text{arith}}$-terms,
   - $(t_1 + t_2)$ is an $\mathcal{L}_{\text{Pres}}$-term.

Example

$\underline{0},\ \underline{1},\ x,\ y,\ ((x + \underline{1}) + y)$	:	$\mathcal{L}_{\text{arith}}$-terms and $\mathcal{L}_{\text{Pres}}$-terms
$((x \times y) \times z),\ (\underline{0} \times (x + y))$	:	$\mathcal{L}_{\text{arith}}$-terms but not $\mathcal{L}_{\text{Pres}}$-terms
$2x,\ y^3,\ \underline{1}/\underline{0},\ +)\times+xy(($	:	neither $\mathcal{L}_{\text{arith}}$-terms nor $\mathcal{L}_{\text{Pres}}$-terms

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Formulas (1/2)

Formulas express a condition for a free variable or a relation among free variables.

Sentences are formal correspondents to the notion of "mathematical statements."

Definition ($\mathcal{L}$-formulas)

1. If t_1 and t_2 are two $\mathcal{L}$-terms, then $t_1 = t_2$ and $t_1 < t_2$ are $\mathcal{L}$-formulas.
2. If φ and ψ are two $\mathcal{L}$-formulas, then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$ $(\varphi \to \psi)$ and $\neg\varphi$ are $\mathcal{L}$-formulas.
3. If φ is an $\mathcal{L}$-formula and x is a variable, then $\exists x\varphi$ and $\forall x\varphi$ are $\mathcal{L}$-formulas.

An $\mathcal{L}$-formula is an $\mathcal{L}$-sentence ($\approx$ "arithmetical sentence") if it has no free variables, i.e., all of the occurrences of a variable in the formula are bound by $\exists$ or $\forall$.

Formulas (2/2)

Example

All of the following are $\mathcal{L}_{\mathsf{Pres}}$-sentences and thus $\mathcal{L}_{\mathsf{arith}}$-sentences.

$$\exists x \exists y((x + y) = \underline{1} \wedge (((x + x) + x) + (y + y)) = (\underline{1} + \underline{1})),$$

$$\forall x \exists y(y < x).$$

Example

Let $\mathrm{Prime}(p)$ be an abbreviation for the $\mathcal{L}_{\mathsf{arith}}$-formula

$$(\underline{1} < p \wedge \forall u \forall v((u < p \wedge v < p) \rightarrow (u \times v) < p)).$$

Then the following are $\mathcal{L}_{\mathsf{arith}}$-sentences.

$$\forall n \exists p(n < p \wedge \mathrm{Prime}(p)),$$

$$\forall n \exists p(n < p \wedge (\mathrm{Prime}(p) \wedge \mathrm{Prime}(p + (\underline{1} + \underline{1})))),$$

$$\forall n(\exists m(((\underline{1} + \underline{1}) \times m) + (\underline{1} + \underline{1})) = n$$
$$\rightarrow \exists p \exists q((\mathrm{Prime}(p) \wedge \mathrm{Prime}(q)) \wedge n = (p + q)).$$

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Satisfaction (1/2)

Here we fix a meaning of our symbols in an obvious way.

Definition (interpretation)

For any $\mathcal{L}_{\text{arith}}$-term t without free variables (i.e., built from $\underline{0}$, $\underline{1}$, $+$ and $\times$), define a natural number $t^{\mathbb{N}}$ by the following rules.

1. $\underline{0}^{\mathbb{N}} = 0$,
2. $\underline{1}^{\mathbb{N}} = 1$,
3. $(t_1 + t_2)^{\mathbb{N}} = t_1^{\mathbb{N}} + t_2^{\mathbb{N}}$,
4. $(t_1 \times t_2)^{\mathbb{N}} = t_1^{\mathbb{N}} \cdot t_2^{\mathbb{N}}$.

Satisfaction (2/2)

Definition (satisfaction relation)

For any $\mathcal{L}_{\text{arith}}$-sentence φ, define $\mathbb{N} \models \varphi$ by the following rules.

1. $\mathbb{N} \models t_1 = t_2 \iff t_1^{\mathbb{N}} = t_2^{\mathbb{N}}$,
2. $\mathbb{N} \models t_1 < t_2 \iff t_1^{\mathbb{N}} < t_2^{\mathbb{N}}$,
3. $\mathbb{N} \models (\varphi \wedge \psi) \iff \mathbb{N} \models \varphi$ and $\mathbb{N} \models \psi$,
4. $\mathbb{N} \models (\varphi \vee \psi) \iff \mathbb{N} \models \varphi$ or $\mathbb{N} \models \psi$,
5. $\mathbb{N} \models (\varphi \to \psi) \iff$ not $\mathbb{N} \models \varphi$ or $\mathbb{N} \models \psi$,
6. $\mathbb{N} \models \neg\varphi \iff$ not $\mathbb{N} \models \varphi$,
7. $\mathbb{N} \models \exists x \varphi(x) \iff$ there exists $n \in \mathbb{N}$ such that $\mathbb{N} \models \varphi(\underbrace{\underline{1} + \cdots + \underline{1}}_{n})$,
8. $\mathbb{N} \models \forall x \varphi(x) \iff \mathbb{N} \models \varphi(\underbrace{\underline{1} + \cdots + \underline{1}}_{n})$ for all $n \in \mathbb{N}$.

Revisiting Incompleteness

Now the precise statement of Gödel's incompleteness theorem is that the following decision problem is undecidable.

Problem (true arithmetic = "first-order theory of the natural numbers")

Input: *an* $\mathcal{L}_{arith}$*-sentence* φ

Question: *Is* $\mathbb{N} \models \varphi$*?*

Thus we have to focus on "weaker" theories in order to develop a deciding algorithm.

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Modifying Our Goal

We are going to construct an algorithm deciding the following problem.

Problem (Presburger arithmetic)

Input: *an $\mathcal{L}_{Pres}$-sentence φ*

Question: *Is $\mathbb{N} \models \varphi$?*

Remark

As a corollary of existence of a deciding algorithm, multiplication $z = x \times y$ is not definable as an $\mathcal{L}_{\text{Pres}}$-formula $\varphi(x, y, z)$.

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The Aim of This Section

We need some techniques from automata theory to construct an algorithm.

So we are going to recall the basics of automata theory quickly.

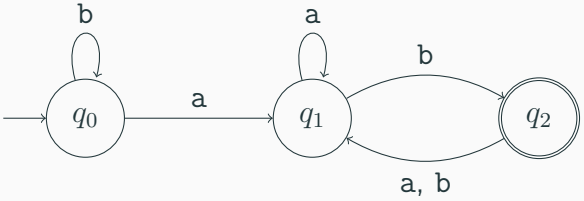
What is a finite automaton?

A finite automaton is a simple mathematical model of the computation, whose storage is extremely limited. Roughly speaking,

$$\text{finite automaton} \approx \text{computer} - \text{its storage}.$$

Finite automata can memorize only finite and fixed amount of information as its current state.

A finite automaton can be visualized as a state diagram like:



Strings

Definition

For a finite set Σ, we write Σ^* for the set of strings over Σ.

Example

If $\Sigma = \{\mathtt{a}, \mathtt{b}\}$, then

$$\Sigma^* = \{\varepsilon, \mathtt{a}, \mathtt{b}, \mathtt{aa}, \mathtt{ab}, \mathtt{ba}, \mathtt{bb}, \mathtt{aaa}, \dots\}$$

where ε is the empty string, whose length is zero.

Formal Definition of DFA

Definition

A deterministic finite automata (DFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ consisting of the following data.

1. a finite set Q of the states,
2. a finite set Σ of the alphabet.
3. transition function $\delta \colon Q \times \Sigma \to Q$,
4. the start state $q_0 \in Q$,
5. a subset $F \subseteq Q$ of the accept states.

Formal Definition of NFA

Definition

An nondeterministic finite automata (NFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ consisting of the following data.

1. a finite set Q of the states,
2. a finite set Σ of the alphabet.
3. transition function $\delta \colon Q \times (\Sigma \cup \{\varepsilon\}) \to \mathcal{P}(Q)$,
4. the start state $q_0 \in Q$,
5. a subset $F \subseteq Q$ of the accept states.

Computation by DFA

Definition

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA. A string $w \in \Sigma^*$ is accepted by M if there are

- a partition $w = w_1 w_2 \cdots w_n$ such that $w_i \in \Sigma$ and
- a finite sequence $r_0, r_1, \ldots, r_n \in Q$ of states

such that

1. $r_0 = q_0$,
2. $\delta(r_i, w_{i+1}) = r_{i+1}$ for $i = 0, 1, \ldots, n-1$,
3. $r_n \in F$.

Otherwise w is rejected by M.

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Definition

Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA. A string $w \in \Sigma^*$ is accepted by N if there are

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Otherwise w is rejected by M.

Regular Languages

Definition (languages)

Let Σ be a finite set. A language over Σ is a subset of Σ^*.

Definition (regular languages)

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA or an NFA. We call the set

$$L(M) = \{\, w \in \Sigma^* \mid M \text{ accepts } w \,\}$$

the language of M.

We say that a language L is recognized by M if $L = L(M)$. A language L is called regular if L is recognized by some DFA.

An Example of DFA

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA defined by:

- $Q = \{q_0, q_1, q_2\}$,
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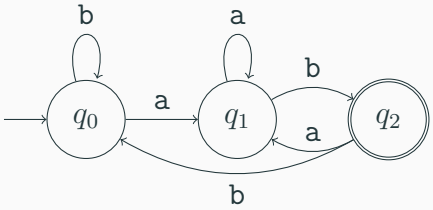

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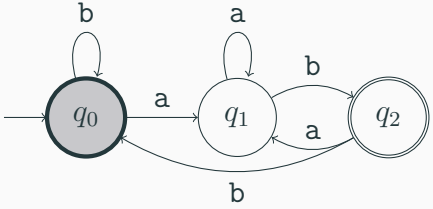

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$$L(M) = \{\, w \in \Sigma^* \mid w \text{ terminates with } \mathtt{ab} \,\}.$$

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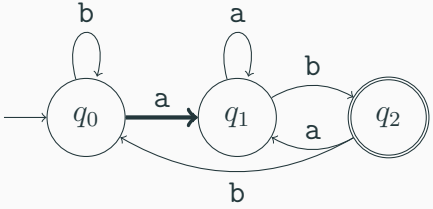

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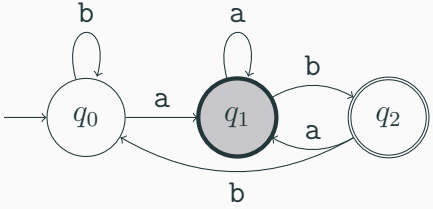

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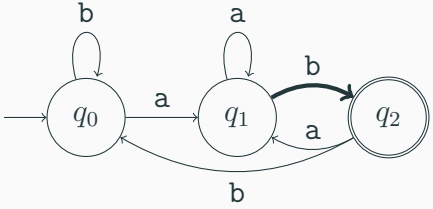

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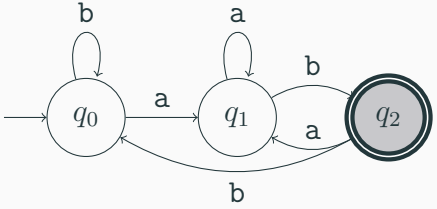

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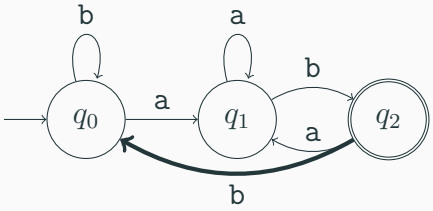

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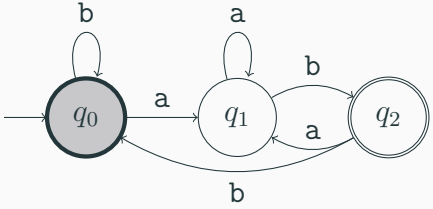

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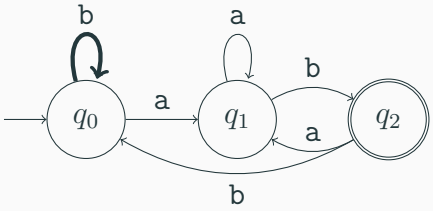

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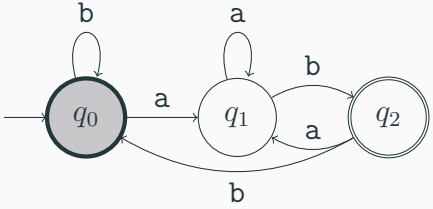

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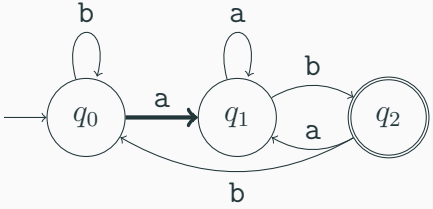

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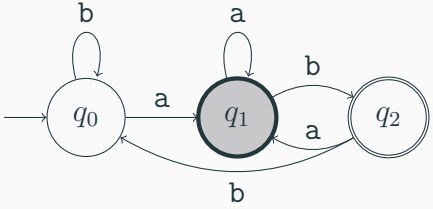

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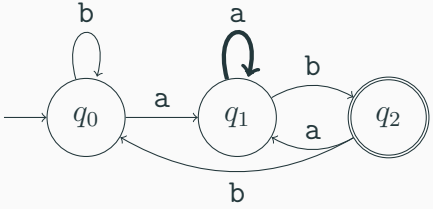

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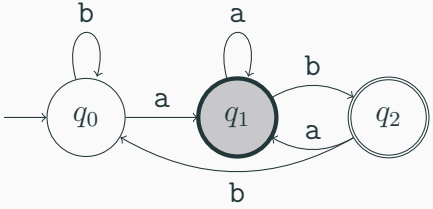

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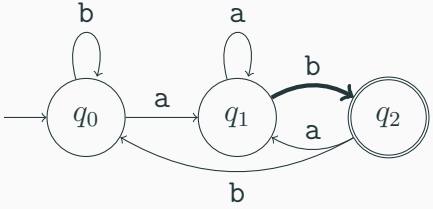

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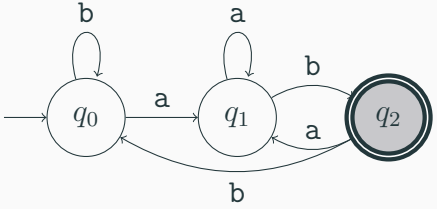

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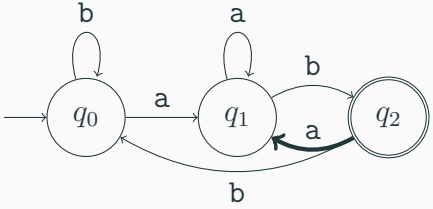

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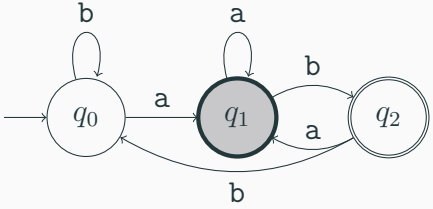

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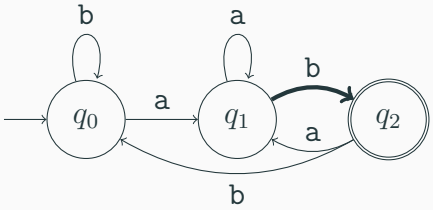

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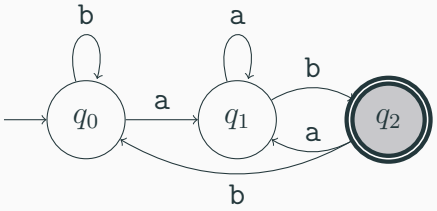

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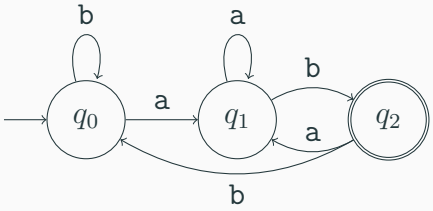

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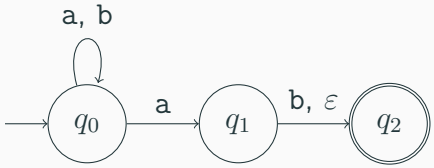

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If we feed a string babb to N as an input string, the computation proceeds as in Figure 2. We see that

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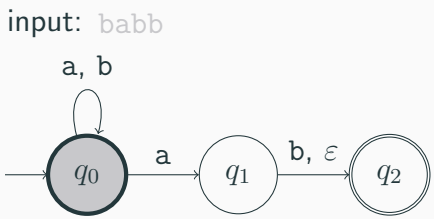

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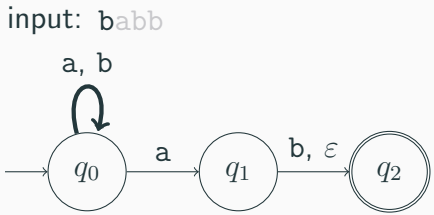

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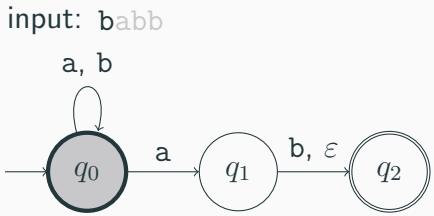

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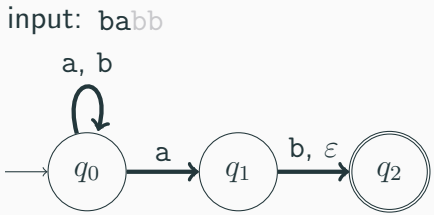

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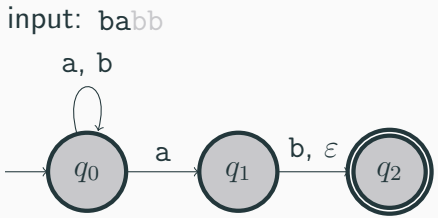

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$$L(N) = \{\, w \in \Sigma^* \mid w \text{ terminates with ab or a} \,\}.$$

An Example of NFA

Let $N = (Q, \Sigma, \delta, q_0, F)$ be an NFA defined by:

- $Q = \{q_0, q_1, q_2\}$,
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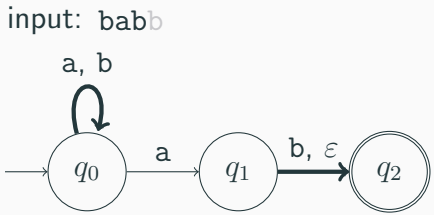

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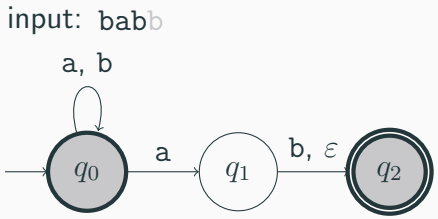

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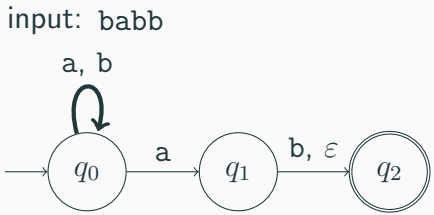

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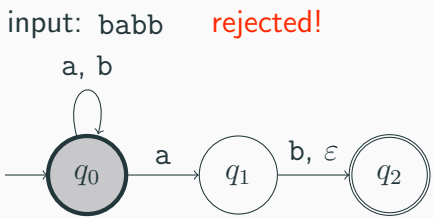

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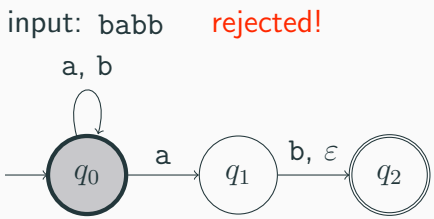

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Subset Construction: Converting NFA into DFA

Theorem

For any NFA N, there exists a DFA M such that $L(N) = L(M)$.

Demonstration

input: abb

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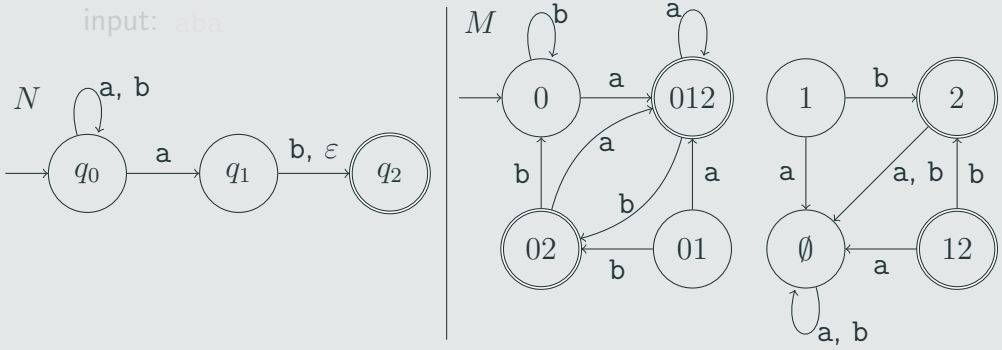

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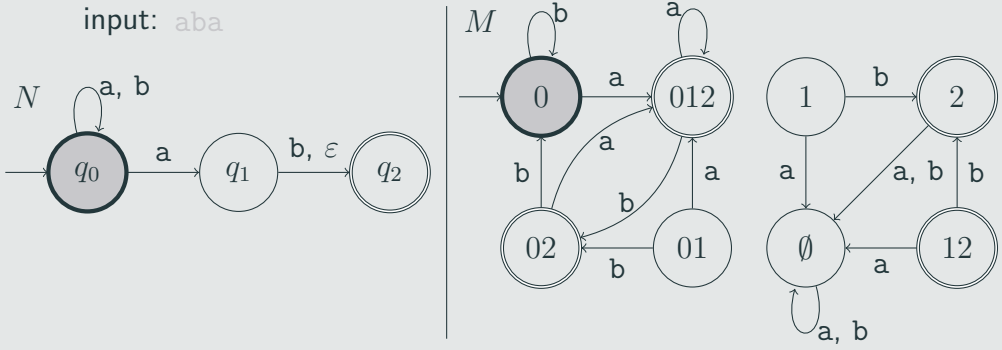

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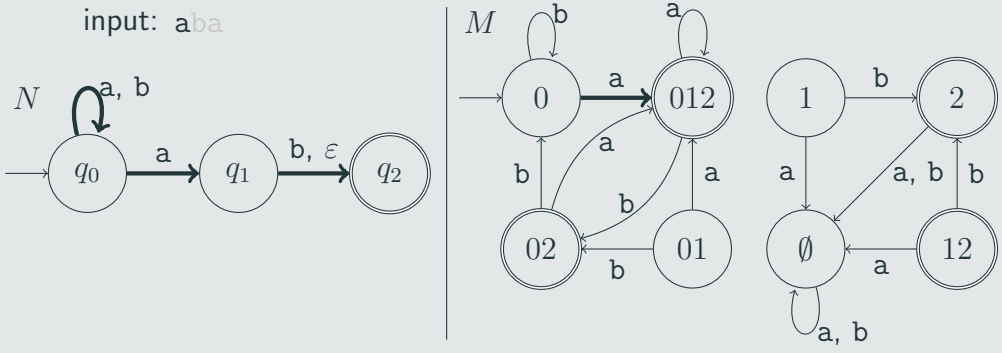

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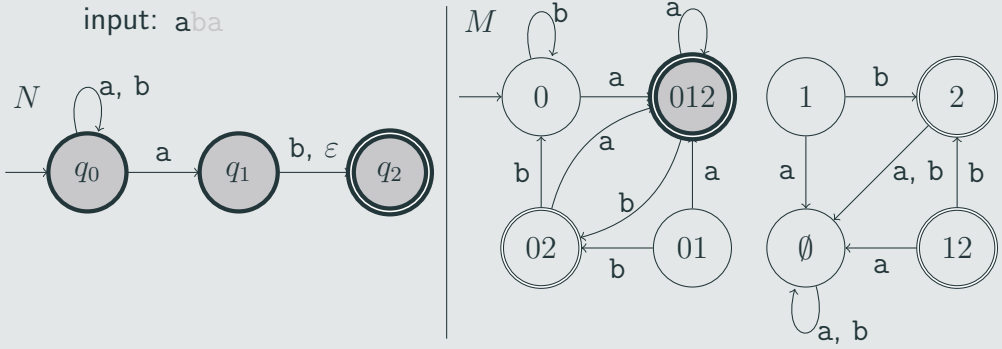

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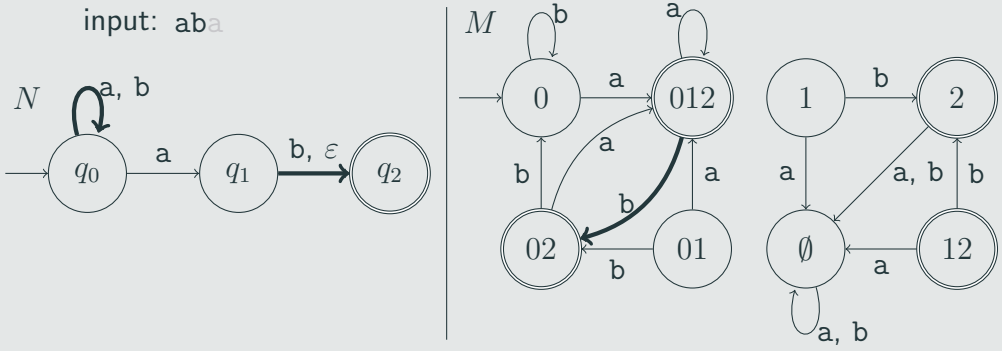

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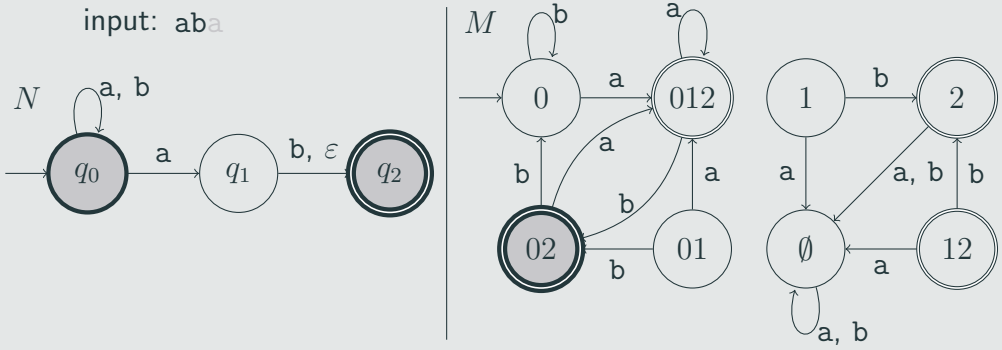

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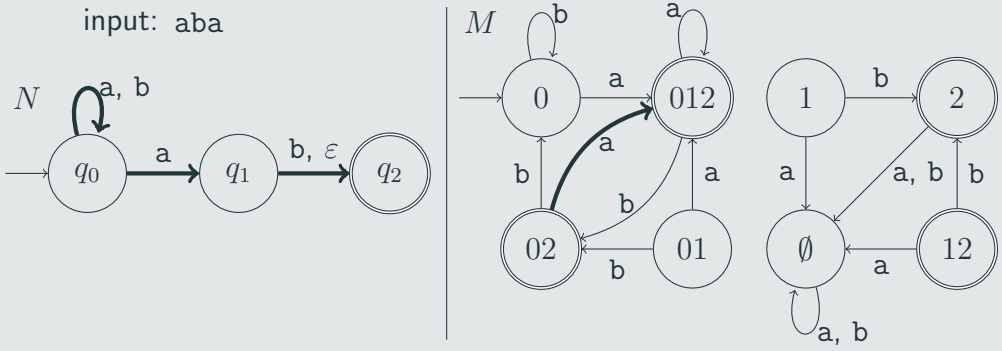

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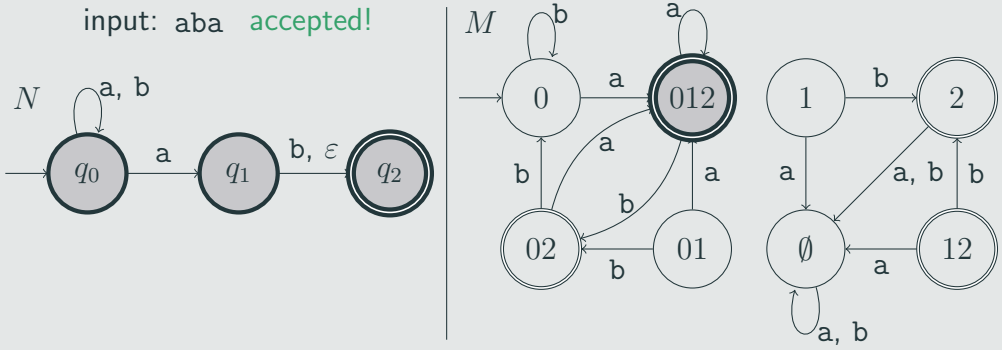

Closure Properties of the Class of the Regular Languages

The class of the regular languages is closed under usual operations.

Theorem

For any regular languages $L_1, L_2 \subseteq \Sigma^*$, the following are also regular.

Complement	$\Sigma^* \setminus L_1$,
Intersection	$L_1 \cap L_2$,
Union	$L_1 \cup L_2$.

Proof Sketch.

Let $M_i = (Q_i, \Sigma, \delta_i, q_i, F_i)$ be a DFA recognizing L_i for $i = 1, 2$. Then

$$M_{\mathsf{C}} = (Q_1, \Sigma, \delta_1, q_1, Q \setminus F_1),$$
$$M_{\mathsf{I}} = (Q_1 \times Q_2, \Sigma, \delta_1 \times \delta_2, (q_1, q_2), F_1 \times F_2),$$
$$M_{\mathsf{U}} = (Q_1 \times Q_2, \Sigma, \delta_1 \times \delta_2, (q_1, q_2), (F_1 \times Q_2) \cup (Q_1 \times F_2))$$

recognizes $\Sigma^* \setminus L_1$, $L_1 \cap L_2$, $L_1 \cup L_2$, respectively. □

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Table of Contents

The Aim of This Section

Now let's build a deciding algorithm for Presburger arithmetic!

Proof (1/10)

Let φ be an $\mathcal{L}_{\text{Pres}}$-sentence. We want to examine the truth value of φ.

First we decompose φ into "simple" parts.

Proof (2/10) — Step 1: Eliminating the Inequality Sign $<$

It is easy to see that

$$t_1 < t_2 \iff \exists x((t_1 + x) + \underline{1}) = t_2.$$

Thus we may assume that φ contians no inequality sign $<$. In other words, φ consists of equations.

Proof (3/10) — Step 2: Decomposing Equations

We can replace each of the equations occurring in φ into a combinations of some equations of the form $(\alpha + \beta) = \gamma$ where α, β, γ are variable symbols or $\underline{0}, \underline{1}$.

Example

One can rewrite the equation

$$(((x + y) + \underline{1}) + x) = (z + z)$$

into

$$\exists u \exists v \exists w ((((x + y) = u \wedge (u + \underline{1}) = v) \wedge (v + x) = w) \wedge (z + z) = w)$$

at the cost of introducing new variable symbols u, v and w.

Proof (4/10) — Step 3: Converting into Prenex Normal Form

Any first-order formula can be converted into the prenex normal form by factoring out $\exists$ and $\forall$.

Example

One can rewrite the $\mathcal{L}_{\mathsf{Pres}}$-sentence

$$\exists x(\exists y(y+y) = x \wedge \forall y \neg x = (y + \underline{1}))$$

into the prenex normal form

$$\exists x \exists y \forall z((y+y) = x \wedge \neg x = (z + \underline{1}))$$

by changing the name of one of the variables y into z.

Thus we may assume that φ is of the form

$$\mathsf{Q}_1 x_1 \mathsf{Q}_2 x_2 \cdots \mathsf{Q}_n x_n \psi(x_1, x_2, \ldots, x_n)$$

where $\psi(x_1, x_2, \ldots, x_n)$ is a Boolean combination of equations of the form $(\alpha + \beta) = \gamma$ and Q_i is either $\exists$ or $\forall$.

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Proof (5/10)

Let ψ_i be a subformula of φ for $i = 0, 1, \ldots, n$ defined by

$$\begin{aligned}
\psi_0 &\equiv \mathsf{Q}_1 x_1 \mathsf{Q}_2 x_2 \cdots \mathsf{Q}_n x_n \psi(x_1, \ldots, x_n) \ (\equiv \varphi), \\
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\psi_{n-1}(x_1, \ldots, x_{n-1}) &\equiv \mathsf{Q}_n x_n \psi(x_1, \ldots, x_n), \\
\psi_n(x_1, \ldots, x_{n-1}, x_n) &\equiv \psi(x_1, \ldots, x_n).
\end{aligned}$$

Proof Idea

For each $i = n, n-1, \ldots, 1, 0$, we are going to construct a finite automaton M_i, which recognizes the values $(a_1, \ldots, a_i) \in \mathbb{N}^i$ satisfying $\mathbb{N} \models \psi_i(a_1, \ldots, a_i)$.

Then we can obtain the truth value of φ by testing whether M_0 accepts ε!

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Proof (6/10) — How to Feed the Number to Automata

For $i = 1, 2, \ldots, n$, let

$$\Sigma_i = \left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 1 \end{bmatrix}, \ldots \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \right\}$$

be the alphabet for M_i, which has 2^i elements. We define $\Sigma_0 = \emptyset$ and $\Sigma_0^* = \{\varepsilon\}$.

In order to know the truth value of $\psi_i(a_1, \ldots, a_i)$, we just need to feed M_i with the binary expansions of $a_1, \ldots, a_i$ from the lowest digit.

Example

Suppose we want to check $\psi_3(3, 4, 13)$. Since $3 = (11)_2$, $4 = (100)_2$ and $13 = (1101)_2$, we will feed M_3 with

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Proof (7/10) — Computing Additions with DFA (1/2)

First we construct M_n that recognizes $\psi_n(a_1, \ldots, a_n)$.
It suffices to check the validity of formulas of the form $\alpha + \beta = \gamma$ thanks to the closure properties of regular languages.
It is the most important case that $x_i + x_j = x_k$ with $i \neq j \neq k \neq i$.

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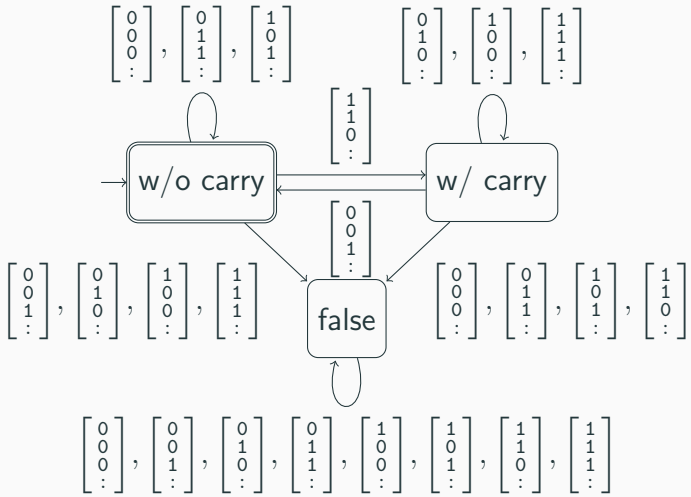

Figure 3: DFA for $a_1 + a_2 = a_3$

Proof (8/10) — Computing Additions with DFA (2/2)

In the other cases,

$\alpha, \beta, \gamma \in \{\underline{0}, \underline{1}\}$	$\Rightarrow$ Accept or reject all of the inputs
$(x_i + \underline{0}) = x_i$	$\Rightarrow$ Accept all of the inputs
$(x_i + x_i) = \underline{1}$ or $(x_i + \underline{1}) = x_i$	$\Rightarrow$ Reject all of the inputs
$(\underline{0} + \underline{0}) = x_i$ or $(x_i + x_i) = \underline{0}$ or $(x_i + x_i) = x_i$ or $(x_i + x_j) = x_j$ $(i \neq j)$	$\Rightarrow$ Check if $a_i = 0$
$(x_i + \underline{0}) = x_j$ $(i \neq j)$	$\Rightarrow$ Check if $a_i = a_j$
$(x_i + x_j) = \underline{0}$ $(i \neq j)$	$\Rightarrow$ Check if $a_i = a_j = 0$
$(\underline{0} + \underline{1}) = x_i$	$\Rightarrow$ Check if $a_i = 1$
$(\underline{1} + \underline{1}) = x_i$	$\Rightarrow$ Check if $a_i = 2$
$(x_i + x_i) = x_j$ $(i \neq j)$	$\Rightarrow$ Similar to the previous page
$(x_i + x_j) = \underline{1}$ $(i \neq j)$ or $(x_i + \underline{1}) = x_j$ $(i \neq j)$	$\Rightarrow$ Similar to the previous page

Proof (9/10) — Manipulating the Quantifier $\exists$

We construct an NFA N_{n-1} based on the DFA M_n. Since $\forall x$ is equivalent to $\neg\exists x\neg$, we may assume that $\psi_{n-1}(a_1, \ldots, a_{n-1}) \equiv \exists x_n \psi(a_1, \ldots, a_{n-1}, x_n)$.

The new NFA N_{n-1} nondeterministically guess the value of x_n. Namely, N_{n-1} can be obtained by truncating the n-th row of each transition in M_n.

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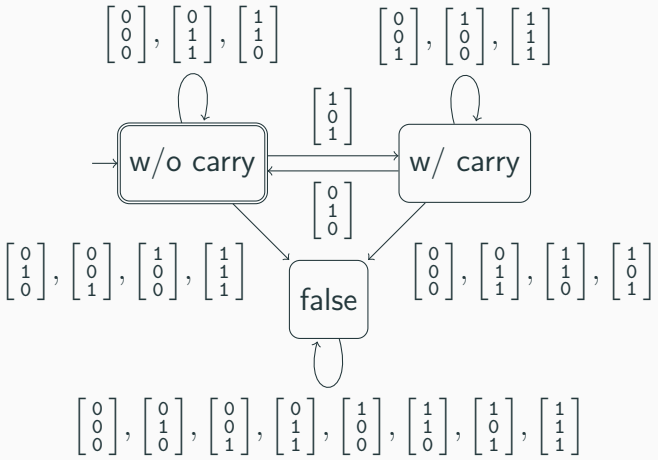

Figure 4: DFA M_3 for $(a_1 + a_3) = a_2$

Proof (9/10) — Manipulating the Quantifier $\exists$

We construct an NFA N_{n-1} based on the DFA M_n. Since $\forall x$ is equivalent to $\neg\exists x\neg$, we may assume that $\psi_{n-1}(a_1, \ldots, a_{n-1}) \equiv \exists x_n \psi(a_1, \ldots, a_{n-1}, x_n)$.

The new NFA N_{n-1} nondeterministically guess the value of x_n. Namely, N_{n-1} can be obtained by truncating the n-th row of each transition in M_n.

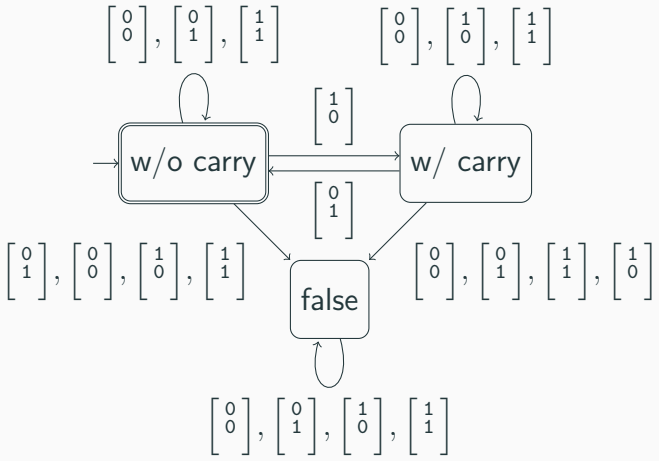

Figure 4: NFA N_2 for $\exists x_3(a_1 + x_3) = a_2$

Proof (10/10) — Summary

Now we can examine the validity of φ in the following way.

1. Eliminate the inequality signs $<$ in φ.
2. Replace each equation in φ with combinations of equations of the form $(\alpha + \beta) = \gamma$.
3. Convert φ into the prenex normal form.
4. Construct the DFA M_n for $\psi_n(a_1, \ldots, a_n)$.
5. Construct the NFA N_{n-1} from M_n by truncating the n-th row.
6. Construct the DFA M_{n-1} from N_{n-1} by the subset construction method.
7. Construct M_0 by repeating this process.
8. Check whether M_0 accepts ε.

Q.E.D.

Proof (10/10) — Summary

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7. Construct M_0 by repeating this process.
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Q.E.D.

Reference

M. Sipser, *Introduction to the Theory of Computation*, Third Edition, Cengage Learning, 2012.